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ON THE CONSTRUCTION AND APPLICATION OF CERTAIN SPECIAL POLARIZATIONS IN NILPOTENT LIE ALGEBRAS

A Dissertation

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* * *

I dedicate this dissertation to my parents, James Donald Moss and Mary Vetrano Moss, who persevered in the (still unfounded) belief that their son would someday finish his education, and also to my wife, Sandra, and to my daughters, Juliette and Genevieve, who did their best to understand a husband and father who would not be understood.

Contents

Abstract

This dissertation arose from efforts to prove the following conjecture, which generalizes to nilpotent Lie groups a weak form of the classical Paley-Wiener theorem for $\mathbb{R}^n$: *Let N be any connected, simply connected nilpotent Lie group with unitary dual $\widehat{N}$, and let $\varphi \in L_c^\infty(N)$. Suppose that there exists a subset $E \subset \widehat{N}$ of positive Plancherel measure such that $\widehat{\varphi}(\pi) = 0$ for all $\pi \in E$, where $\widehat{\varphi}(\pi)$ is the operator-valued Fourier transform of φ. Then $\varphi = 0$ almost everywhere on N.* The writer has been able to prove a slightly weakened form of the conjecture for a large subclass of nilpotent Lie groups, and the conjecture itself for several interesting examples that lie outside this subclass. Chapter 3 contains these proofs, which make use of certain special polarizations (maximal subordinate subalgebras) of the Lie algebras there considered. Chapter 2 explains how to construct such polarizations in *any* nilpotent Lie algebra. Chapter 1 provides background for the work undertaken in Chapters 2 and 3.

CHAPTER 1

Introduction and Background

§ 1.1 A Conjecture and its Antecedents

This dissertation arose from efforts to prove the following conjecture, which generalizes to nilpotent Lie groups a weak form of the classical Paley-Wiener theorem for $\mathbb{R}^n$:

1.1 Conjecture. *Let N be any connected, simply connected nilpotent Lie group with unitary dual $\widehat{N}$, and let φ be a compactly supported, measurable, essentially bounded function on N (that is, $\varphi \in L_c^\infty(N)$). Suppose that there exists a subset $E \subset \widehat{N}$ of positive Plancherel measure such that*

$$\widehat{\varphi}(\pi) = 0 \quad \textit{for all} \quad \pi \in E,$$

where $\widehat{\varphi}(\pi)$ is the operator-valued Fourier transform of φ. Then $\varphi = 0$ almost everywhere on N.

The Paley-Wiener theorem rests upon the fact that if $\varphi \in L_c^\infty(\mathbb{R}^n)$, then $\widehat{\varphi}$, the ordinary Fourier transform of φ, has an *entire* extension to $\mathbb{C}^n$. This fact permits one to conclude that $\widehat{\varphi}$ is identically zero if it vanishes on a set of positive measure. But there exists no natural complex structure which corresponds to $\widehat{N}$ in the way that $\mathbb{C}^n$ corresponds to $\mathbb{R}^n$, and so there is a fundamental obstacle to the proof of the proposed conjecture. Nonetheless, the writer has been able to prove a slightly weakened form of the conjecture for a large subclass of nilpotent Lie groups, and the conjecture itself for several interesting examples that lie outside this subclass. Chapter 3 contains these proofs, which make use of certain special polarizations (maximal subordinate subalgebras) of the Lie algebras there considered. Chapter 2 explains how to construct such polarizations in any nilpotent Lie algebra.

The writer first encountered a version of Conjecture 1.1 in a paper of Scott and Sitaram, "Some remarks on the Pompeiu problem for groups" [11]. Lemma 4.1 of that paper amounts to the contrapositive of Conjecture 1.1 for the $(2n+1)$-dimensional Heisenberg group, the prototypical non-abelian nilpotent Lie group. Scott and Sitaram ask, but do not answer, the question whether their result holds for arbitrary simply connected nilpotent Lie groups.

Corwin and Greenleaf, in their paper "Fourier transforms of smooth functions on certain nilpotent Lie groups" [1], prove a theorem which implies a slightly weakened form of Conjecture 1.1 for a special subclass of nilpotent Lie groups, viz., those for which a *single ideal* in the corresponding Lie algebra polarizes all general position linear functionals in the dual of the algebra. Since such groups are included among those for which the writer has been able to prove the weakened form of the conjecture, more will be said about them in Chapter 3 (see §3.2).

§ 1.2 A Little Nilpotent Lie Theory†

A Lie algebra $\mathfrak{n}$ is called 'nilpotent' if its descending central series is finite, where

$$\mathfrak{n}^{(1)} = \mathfrak{n},$$

$$\mathfrak{n}^{(k+1)} = [\mathfrak{n}, \mathfrak{n}^{(k)}] = \mathbb{R}\text{-span}\{[X,Y] : \ X \in \mathfrak{n}, Y \in \mathfrak{n}^{(k)}\},$$

defines that series inductively. If $\mathfrak{n}^{(k+1)} = 0$, but $\mathfrak{n}^{(k)} \neq 0$, then $\mathfrak{n}$ is said to be 'k-step nilpotent', and $\mathfrak{n}^{(k)} \neq 0$ shows that the center of the algebra is non-trivial. The Birkhoff Embedding Theorem ([3], p. 7) shows that every nilpotent Lie algebra is isomorphic to a subalgebra of the algebra of all $j \times j$ strictly upper triangular matrices for some j. The corresponding connected, simply connected nilpotent Lie group N, which is obtained by exponentiating the algebra $\mathfrak{n}$, may then be embedded as a subgroup of the group of upper triangular $j \times j$ matrices with 1's on the diagonal.

If G is any connected Lie group with exponential map $\exp : \mathfrak{g} \to G$, the product operation

$$X * Y = \log(\exp X \cdot \exp Y), \quad X, Y \in \mathfrak{g},$$

defines an analytic function near $X = Y = 0$ which is given by a universal power series involving bracket products, or commutators, of the Lie algebra $\mathfrak{g}$. We shall use this so-called Campbell-Baker-Hausdorff (C-B-H) formula in Chapter 3. For

† For standard facts of nilpotent Lie theory, the writer relies almost entirely upon the recent text of Corwin and Greenleaf, *Representations of nilpotent Lie groups and their applications* [3].

the reader's amusement, we give the general expression and then the first few terms (here, $(\mathrm{ad}\,A)B = [A, B]$):

$$X * Y = \sum_{n>0} \frac{(-1)^{n+1}}{n} \sum_{\substack{p_i+q_i>0 \\ 1\le i\le n}} \frac{(\sum_{i=1}^{n}(p_i+q_i))^{-1}}{p_1!q_1!\cdots p_n!q_n!}$$

$$\times (\mathrm{ad}\,X)^{p_1}(\mathrm{ad}\,Y)^{q_1}\cdots(\mathrm{ad}\,X)^{p_n}(\mathrm{ad}\,Y)^{q_n-1}\,Y$$

$$= X + Y + \tfrac{1}{2}[X,Y] + \tfrac{1}{12}[X,[X,Y]] - \tfrac{1}{12}[Y,[X,Y]]$$

$$- \tfrac{1}{48}[Y,[X,[X,Y]]] - \tfrac{1}{48}[X,[Y,[X,Y]]]$$

$$+ (\text{commutators in five or more terms}).$$

(If $q_n = 0$, the term in the sum is $\ldots(\mathrm{ad}\,X)^{p_n-1}$; if $q_n > 1$, or if $q_n = 0$ and $p_n > 1$, then the term is zero.) If G is a general connected Lie group, the C-B-H formula allows one to reconstruct the group locally (i.e., near the identity) from knowledge of the brackets of the algebra $\mathfrak{g}$. But if N is a connected, simply connected nilpotent Lie group, the situation is nicer. In fact, the C-B-H series is finite, giving global polynomial laws for the group multiplication. Also, $\exp : \mathfrak{n} \to N$ is an analytic diffeomorphism *onto* N, and the C-B-H formula holds for all $X, Y \in \mathfrak{n}$ ([3], p. 13).

In Chapters 2 and 3 we shall take for granted an acquaintance with two kinds of bases in a nilpotent Lie algebra $\mathfrak{n}$. The set $\mathcal{B} = \{X_1, \ldots, X_n\}$ is a *strong Malcev basis* for $\mathfrak{n}$ if $\mathfrak{n}_j = \mathbb{R}\text{-span}\{X_1, \ldots, X_j\}$ is an ideal for each $1 \le j \le n$. The set $\mathcal{B}_w = \{X_1, \ldots, X_n\}$ is a *weak Malcev basis* for $\mathfrak{n}$ if $\mathfrak{n}_j = \mathbb{R}\text{-span}\{X_1, \ldots, X_j\}$ is a subalgebra for each $1 \le j \le n$. Every nilpotent Lie algebra may be given bases of either kind ([3], p. 10). As we shall see in a few moments, strong Malcev bases play a special role in nilpotent representation theory.

If $x, y \in N$, then the map $\alpha_x : y \to xyx^{-1}$ is an inner automorphism of N, and its differential at the identity element, $\mathrm{Ad}\, x = \mathrm{d}(\alpha_x)_e : \mathfrak{n} \to \mathfrak{n}$, is an automorphism of N. The action $N \times \mathfrak{n} \to \mathfrak{n}$ given by $(x, Y) \to (\mathrm{Ad}\, x)Y$ is called the *adjoint action* of N ([3], p. 12). We have the formula

$$\exp\big((\mathrm{Ad}\, x)Y\big) = \alpha_x(\exp Y), \text{ for all } x \in N,\ Y \in \mathfrak{n},$$

from which it follows that, if $x = \exp X$,

$$\alpha_{\exp X}(\exp Y) = \exp\big(\mathrm{Ad} \exp X)Y\big).$$

Two other formulas that we shall find useful are

$$(\mathrm{Ad} \exp X)Y = e^{\mathrm{ad}\, X}(Y) = \sum_{k=0}^{\infty} \frac{1}{k!}(\mathrm{ad}\, X)^k Y, \text{ for all } X, Y \in \mathfrak{n},$$

(which sum, because of nilpotence, possesses only finitely many non-zero terms) and

$$(\mathrm{Ad} \exp X)Y = X * Y * (-X).$$

If $\mathfrak{n}^*$ denotes the linear dual of $\mathfrak{n}$, then N acts on $\mathfrak{n}^*$ by the contragredient of the adjoint map, the *coadjoint map* Ad^*:

$$\big((\mathrm{Ad}^*\, x)\ell\big)Y = \ell\big((\mathrm{Ad}\, x^{-1})Y\big), \quad Y \in \mathfrak{n}, \quad \ell \in \mathfrak{n}^*, \text{ and } x \in N.$$

Kirillov, in his seminal 1962 paper [5], revealed the importance of this map in the representation theory of nilpotent Lie groups, showing that the set $\widehat{N}$ of (equivalence classes) of irreducible unitary representations of N is naturally parametrized by the orbits of $\mathfrak{n}^*$ under the coadjoint action of N (see §1.3 below).

Our work in Chapter 2 will use the differential of the coadjoint map at the identity in N. This map, denoted $\mathrm{ad}^* : \mathfrak{n} \to \mathrm{End}(\mathfrak{n}^*)$, is defined by

$$\big((\mathrm{ad}^* X)\ell\big)(Y) = \ell([Y,X]) = \ell\big(\mathrm{ad}(-X)Y\big), \quad X, Y \in \mathfrak{n}, \text{ and } \ell \in \mathfrak{n}^*,$$

as one shows by evaluating at $t = 0$ the derivative with respect to t of the series

$$\mathrm{Ad}(\exp tX) = e^{\mathrm{ad}\, tX} = \sum_{k=0}^{\infty} \frac{t^k}{k!}(\mathrm{ad}\, X)^k, \text{ for all } X \in \mathfrak{n}.$$

Using ad^*, the coadjoint action of N on $\mathfrak{n}^*$ may be written, for all $X \in \mathfrak{n}$,

$$\mathrm{Ad}^*\big(\exp tX\big)(\ell) = e^{\mathrm{ad}^*\, tX}(\ell) = \sum_{k=1}^{\infty} \frac{t^k}{k!}(\mathrm{ad}^* X)^k(\ell).$$

Again, the series has only finitely many non-zero terms in the nilpotent context.

If $\ell \in \mathfrak{n}^*$, the stabilizer subgroup of ℓ under the coadjoint action of N is denoted by

$$R_\ell = \{x \in N : \ (\mathrm{Ad}^*\, x)\ell = \ell\}.$$

The Lie algebra of R_ℓ is an important subalgebra of $\mathfrak{n}^*$, the *radical* of ℓ:

$$\mathfrak{r}_\ell = \{X \in \mathfrak{n} : \ (\mathrm{ad}^* X)\ell = 0\}.$$

The coadjoint orbit $\mathcal{O}_\ell = (\mathrm{Ad}^* N)\ell$ is homeomorphic to the homogeneous space N/R_ℓ and the dimension of the orbit is given by $\dim \mathfrak{n} - \dim \mathfrak{r}_\ell$ ([3], p. 26). This dimension is always even, as can be seen from the fact that $\mathfrak{r}_\ell$ is the radical of an antisymmetric bilinear form, to wit, $B_\ell : \mathfrak{n} \times \mathfrak{n} \to \mathbb{R}$, defined by

$$B_\ell(X,Y) = \ell([X,Y]), \quad X, Y \in \mathfrak{n}.$$

We shall become better acquainted with B_ℓ in Chapter 2, but in the meantime let us note a couple of important facts about any such form B (see [3], p. 27).

If V is a real vector space with an antisymmetric bilinear form B, its *isotropic* subspaces W are those such that $B(w_1, w_2) = 0$ for all $w_1, w_2 \in W$. Linear algebra tells us that maximal isotropic subspaces always exist, and all have the same (even) dimension

$$\tfrac{1}{2}\dim(V/\operatorname{rad} B) + \dim(\operatorname{rad} B) = \tfrac{1}{2}(\dim\, V + \dim\, \operatorname{rad} B),$$

where $\operatorname{rad} B = \{v_1 \in V :\ B(v_1, v_2) = 0, \ \text{ for all } \ v_2 \in V\}$. If we let $r = \dim\, \operatorname{rad} B$ and $k = \frac{1}{2}\dim(V/\operatorname{rad} B)$, we see that $2k = n - r$ is the dimension of each of these subspaces, where $n = \dim\, \mathfrak{n}$.

Now if $V = \mathfrak{n}$, $\ell \in \mathfrak{n}^*$ and $B = B_\ell$, then $\operatorname{rad} B_\ell = \mathfrak{r}_\ell$ and there are maximal isotropic subspaces $\mathfrak{m}_\ell$ for B_ℓ which are also *subalgebras* of $\mathfrak{n}$. Such subalgebras are said to be *polarizing* for ℓ, and are usually called *polarizations*. They are important in the representation theory of Lie algebras and Lie groups for two reasons. In the first place, their isotropy insures that $\ell([\mathfrak{m}_\ell, \mathfrak{m}_\ell]) = 0$, so the character $\chi(\exp\, \mathfrak{m}_\ell) = e^{2\pi i\,\ell(\mathfrak{m}_\ell)}$ defines a 1-dimensional representation of the subgroup $M_\ell = \exp\, \mathfrak{m}_\ell$. In the second place, their maximal isotropy insures that χ induces to an *irreducible unitary* representation of N. Chapter 2 below is devoted to the construction of certain special polarizations in nilpotent Lie algebras. But before we get there, we make the briefest of sketches of the subject just broached.

§ 1.3 A Little Less Nilpotent Representation Theory

A *unitary representation* of a locally compact group G is a strong operator continuous homomorphism π of G into the group $\mathcal{U}(\mathcal{H}_\pi)$ of unitary operators on a Hilbert space $\mathcal{H}_\pi$, where the strong operator continuity means that

$$\| \pi_{g_n}\xi - \pi_g\xi \| \to 0 \quad \text{as} \quad g_n \to g, \quad \text{for all} \quad \xi \in \mathcal{H}_\pi.$$

Such a representation is said to be *irreducible* if $\mathcal{H}_\pi$ contains no proper, closed, $\pi(G)$-invariant subspaces. The set of equivalence classes of irreducible unitary representations of G is denoted by $\widehat{G}$.

If one already has in hand a representation σ of a closed subgroup $K \subseteq G$ in a Hilbert space $\mathcal{H}_\sigma$, then a representation π of the full group G in a new Hilbert space $\mathcal{H}_\pi$ may be obtained by the method of *induction*. The induced representation is denoted $\pi = \mathrm{Ind}(K\uparrow G, \pi)$, and, for unimodular groups (among which are to be found all nilpotent Lie groups), the construction goes as follows (see [3], pp. 39ff.). Let $\mathcal{H}_\pi$ be the Hilbert space of all Borel measurable vector-valued functions $f : G \to \mathcal{H}_\sigma$ that are

(i) covariant like σ along K-cosets : $f(kg) = \sigma(k)\big(f(g)\big)$,

(ii) square-summable, *i.e.*, $\int_{K\backslash G} \| f(g) \|^2_{\mathcal{H}_\sigma} \; d\dot{g} < \infty$, where $d\dot{g}$ is right-invariant measure on $K\backslash G$.

Now the map $g \to \|f(g)\|^2_{\mathcal{H}_\sigma}$ is constant on $K\backslash G$-cosets, and so too is the map $g \to \langle f_1(g), f_2(g) \rangle_{\mathcal{H}_\sigma}$. The inner product on $\mathcal{H}_\pi$ is

$$\langle f_1(g), f_2(g) \rangle_{\mathcal{H}_\pi} = \int_{K\backslash G} \langle f_1(g), f_2(g) \rangle_{\mathcal{H}_\sigma} \; d\dot{g},$$

and $\mathcal{H}_\pi$ is complete with respect to this inner product. The *induced representation* π is defined by a right action of G on functions $f \in \mathcal{H}_\sigma$:

$$\pi(x)f(g) = f(gx) \text{ for all } x \in G.$$

By the right-invariance of the measure $d\dot{g}$, this action is a unitary operation. Its strong operator continuity follows from the fact that the set of functions

$$\mathcal{H}_{\pi,c} = \{f \in \mathcal{H}_\pi : f \text{ is continuous and } \mathrm{supp}(f) \text{ has compact image in } K\backslash G\}$$

is L^2-dense in $\mathcal{H}_\pi$.

To illustrate these concepts, let K be any closed subgroup of G such that $K\backslash G$ has an invariant measure $d\dot{g}$. If the representation on K is the trivial representation $\sigma = 1$ on $\mathcal{H}_\sigma = \mathbb{C}$, then the functions in $\mathcal{H}_\pi$ are scalar-valued, constant on $K\backslash G$-cosets and there is an isometry of $\mathcal{H}_\pi \cong L^2(K\backslash G, \mathbb{C})$ which carries $\pi(g)$ to the right action $R_g f(\xi) = f(\xi \cdot g)$, where $\xi = Kx \in K\backslash G$. In particular, if $K = (e)$ and $d\dot{g} =$ right Haar measure on G, then $\mathcal{H}_\pi = L^2(G)$ and $\pi = \mathrm{Ind}(K\uparrow G, 1)$ is the right regular representation of G ([3], p.40).

In the house of induced representations there are many models. Denote by $p : G \to K\backslash G$ the natural quotient map; then there always exists a Borel cross-section map λ to p such that $p \circ \lambda = \mathrm{id}$. If we let $\Sigma = \lambda(K\backslash G) \subseteq G$, then any $f \in \mathcal{H}_\pi$ is completely determined by its values on the transversal Σ (because of the covariance mentioned above). The map $f \to f \circ \lambda$, which takes functions in $\mathcal{H}_\pi$ to functions on $K\backslash G$ with values in $\mathcal{H}_\sigma$, is an isometry from $\mathcal{H}_\pi$ to $L^2(K\backslash G, \mathcal{H}_\sigma)$. Hence the unitary action of G on $\mathcal{H}_\pi$ can be realized in $L^2(K\backslash G, \mathcal{H}_\sigma)$ as a cocycle action involving translation in the base space $K\backslash G$ and concurrent action on the values in $\mathcal{H}_\sigma$. This description, though somewhat opaque, accurately sums up what is happening when we choose a cross-section $\lambda : K\backslash G \to G$ and compute the (measurable) splitting of a typical group element $w = k(w) \cdot s(w) \in K \cdot \Sigma = K \cdot \lambda(K\backslash G)$. If $x \in G$ and $f \in \mathcal{H}_\sigma$, we have

$$\pi(g)f(x) = f(x \cdot g) = f\big(k(xg) \cdot s(xg)\big) = \sigma_{k(xg)}\big[f\big(s(xg)\big)\big].$$

So, identifying $\mathcal{H}_\pi \cong L^2(K\backslash G, \mathcal{H}_\sigma)$ by sending $f \to \tilde{f}(\zeta) = f \circ \lambda(\zeta)$, we have

$$\pi(g)\tilde{f}(\zeta) = \sigma_{k\left(\lambda(\zeta)g\right)}\big[\tilde{f}(\zeta \cdot g)\big] \quad \forall\, \zeta \in K\backslash G, \forall\, g \in G,$$

because $s\big(\lambda(\zeta) \cdot g\big) = s\big(K \cdot \lambda(\zeta)g\big) = s(\zeta \cdot g)$ is a well-defined element of the transversal Σ.

We shall use such a computation at the outset of Chapter 3 (§3.1), but of course only in the more restricted nilpotent context. Up to unitary equivalence, all irreducible representations of a nilpotent Lie group are induced from monomial representations (characters) $\chi : M \to S^1$ of subgroups $M = \exp(\mathfrak{m})$, where $\mathfrak{m}$ is a polarization. In more detail, let $\ell \in \mathfrak{n}^*$, the linear dual of $\mathfrak{n}$. Then, as we saw in §1.2, N acts on $\mathfrak{n}^*$ by the coadjoint action $\mathrm{Ad}^*(N)$. If $\ell \in \mathfrak{n}^*$, choose a polarization $\mathfrak{m}$ for ℓ and let $M = \exp(\mathfrak{m})$. Then the map $M \to S^1$ defined by

$$\chi_{\ell,M}(\exp Y) = e^{2\pi i \ell(Y)}, \quad Y \in \mathfrak{m},$$

is a one-dimensional representation of M, since $\ell\big([\mathfrak{m},\mathfrak{m}]\big) = 0$. Hence we may form the induced representation $\pi_{\ell,M} = \mathrm{Ind}(M{\uparrow}G, \chi_{\ell,M})$, after the fashion explained a moment ago.

The following results of Kirillov [5] describe the unitary dual $\widehat{N}$ in terms of these induced represetations (see [3], pp. 45–46).

1.3.1 Theorem. *Let $\ell \in \mathfrak{n}^*$. Then there exists a polarization $\mathfrak{m}$ for ℓ, and the induced representation $\pi_{\ell,M}$ is irreducible.*

1.3.2 Theorem. *Let $\ell \in \mathfrak{n}^*$, and let $\mathfrak{m}$, $\mathfrak{m}'$ be two polarizations for ℓ. Then $\pi_{\ell,M} \cong \pi_{\ell,M'}$. (Hence we may write π_ℓ for $\pi_{\ell,M}$ if we are interested only in equivalence classes of unitary representations.)*

1.3.3 Theorem. *Let π be any irreducible unitary representation of N. Then there is an $\ell \in \mathfrak{n}^*$ such that $\pi_\ell \cong \pi$.*

1.3.4 Theorem. *Let $\ell, \ell' \in \mathfrak{n}^*$. Then $\pi_\ell \cong \pi_{\ell'} \iff \ell$ and ℓ' are in the same $\mathrm{Ad}^*(N)$-orbit in $\mathfrak{n}^*$.*

We shall not concern ourselves with the proofs of these theorems. However, it should be understood that the theorems lie at the heart of nilpotent representation theory and are taken for granted in Chapters 2 and 3 below.

§ 1.4 Parametrization of Coadjoint Orbits

The Kirillov theorems highlight the significance of coadjoint orbits in the representation theory of nilpotent Lie groups. But so far we have said very little about the orbits themselves. The Chevalley-Rosenlicht theorem permits us to say more. We shall state a special case—the nilpotent case—of this theorem. (For the statement and proof of a more general version of this theorem, see [3], pp. 82–83.)

1.4 Theorem (Chevalley-Rosenlicht). *Let N be a connected, simply connected nilpotent Lie group, and let $\mathrm{Ad}^*(N)\ell$ denote the orbit of $\ell \in \mathfrak{n}^*$ under the coadjoint action of N. Then there exist vectors $X_1, \ldots, X_w \in \mathfrak{n}$, the Lie algebra of N, such that*

$$\mathrm{Ad}^*(N)\,\ell = \left\{\mathrm{Ad}^*\big(\exp t_1X_1 \cdots \exp t_wX_w\big)\,\ell :\ t_1, \ldots, t_w \in \mathbb{R}\right\}.$$

The map

$$P(t_1, \ldots, t_w) = \mathrm{Ad}^*\big(\exp t_1X_1 \cdots \exp t_wX_w\big)\,\ell$$

is a diffeomorphism between $\mathbb{R}^w$ and the orbit $\mathrm{Ad}^*(N)\,\ell$, and the orbit is a closed submanifold of $\mathfrak{n}^*$. In fact, let $\{X_1^*, \ldots, X_n^*\}$ be a basis for $\mathfrak{n}^*$ such that $\mathbf{n}_j^* = \mathbb{R}\text{-span}\{X_{j+1}^*, \ldots, X_n^*\}$ is $\mathrm{Ad}^*(N)$ stable for all j, and define polynomials $P_1, \ldots, P_n$ such that

$$\begin{aligned}\mathrm{Ad}^*\big(\exp t_1X_1 \cdots \exp t_wX_w\big)\,\ell &= \sum_{j=1}^{n} P_j(t_1, \ldots, t_w)X_j^* \\ &= P(t_1, \ldots, t_w).\end{aligned}$$

Then there exist disjoint sets of indices $J \cup D = \{1, \ldots, n\}$ *such that the set* $J = \{j_1 < \cdots < j_w\}$, *and the polynomial* P_j *depends only on those variables* t_i *with* $j_i \leq j$. *Moreover,*

$$P_{j_i} = t_i + (\text{a polynomial in } t_1, \ldots, t_{i-1}) \quad \text{for} \quad 1 \leq i \leq w.$$

A basis $\{X_1^*, \ldots, X_n^*\}$ with the property that $\mathrm{Ad}^*(N)(\mathfrak{n}_j^*) \subseteq \mathfrak{n}_{j+1}^*$ is called a Jordan-Hölder basis for $\mathfrak{n}^*$. Such bases always exist by the classical theorem of Engel ([3], p.4).

A question we should address before embarking on Chapter 2 is this: What is the provenance of the index set J? How is it determined? The answer requires a glance at part of the proof of the theorem. Let $\mathfrak{n}_j^* = \{X_{j+1}^*, \ldots, X_n^*\}$ and $\mathfrak{n}_n^* = (0)$; by hypothesis, $\mathfrak{n}_j^*$ is $\mathrm{Ad}^*(N)$-stable. Let

$$\begin{aligned} \mathfrak{n}_j &= \{X \in \mathfrak{n} : \ \mathrm{ad}^*(X)(\ell) \in \mathfrak{n}_j^*\} \\ &= \{X \in \mathfrak{n} : \ \mathrm{ad}^*(X)(\ell) = 0 \bmod \mathfrak{n}_j^*\}. \end{aligned}$$

Then $\mathfrak{n}_0 = \mathfrak{n}_1 = \mathfrak{n}$ (because N acts unipotently†), and $\mathfrak{n}_0 \supseteq \mathfrak{n}_1 \supseteq \cdots \supseteq \mathfrak{n}_n$. What is $\mathfrak{n}_j$ algebraically? It is the annihilator of $p(\ell) \in \mathfrak{n}^*/\mathfrak{n}_j^*$ under the quotient coadjoint action of $\mathfrak{n}$ on $\mathfrak{n}^*/\mathfrak{n}_j^*$, where $p : \mathfrak{n}^* \to \mathfrak{n}^*/\mathfrak{n}_j^*$ is the natural quotient map. The set $J = \{j_1, \ldots, j_w\}$ is then the set of indices for which $\mathfrak{n}_{j-1} \supsetneqq \mathfrak{n}_j$ $(j_1 \geq 2)$, and so J is the set of 'jump' indices, recording where the orbits of the quotient coadjoint action increase their dimension as we travel up (down?) the Jordan-Hölder series for $\mathfrak{n}^*$.

† As we saw above, every nilpotent Lie group is isomorphic to an upper triangular matrix group, each element of which has 1's on the diagonal. A 'unipotent' action, in matrix terms, is an action by such a matrix.

Recall now that a set $U \subseteq \mathfrak{n}^*$ is called *Zariski-open* if it is a union of sets $\{\ell \in \mathfrak{n}^* : P(\ell) \neq 0\}$, where P is a polynomial. If $\{X_1^*, \ldots, X_n^*\}$ is a Jordan-Hölder basis for $\mathfrak{n}^*$ and the index sets J and D are determined as in the Chevalley-Rosenlicht theorem, then the set $U \subseteq \mathfrak{n}^*$ of linear functionals ℓ for which the dimensions of quotient coadjoint orbits in $\mathfrak{n}^*/\mathfrak{n}_j^*$ are *as large as possible* for each $1 \leq j \leq n$, is an $\mathrm{Ad}^*(N)$-invariant Zariski-open set. The coadjoint orbits in U are the so-called 'generic' or 'general position' orbits.

Hence we see that the definition of 'generic' is *essentially basis-dependent*. If we change the Jordan-Hölder basis for $\mathfrak{n}^*$, we also in general change the set of generic orbits. And since the linear dual basis of a Jordan-Hölder basis is a strong Malcev basis for $\mathfrak{n}$, the same comment applies, *mutatis mutandis*, to such a basis. Since our work in Chapter 2 involves fixing a strong Malcev basis for $\mathfrak{n}$ and using matrix techniques to get what we want, we thought that the basis-dependence inherent in the concept of genericity should be noted at the outset.

CHAPTER 2

Constructing Special Polarizations

§2.1 A Little More Background

Let $\mathcal{B} = \{X_1, \ldots, X_n\}$ be a strong Malcev basis for the Lie algebra $\mathfrak{n}$ of a connected, simply connected nilpotent Lie group N. Then the dual basis $\mathcal{B}^* = \{X_1^*, \ldots, X_n^*\}$ is a Jordan-Hölder basis for $\mathfrak{n}^*$, the linear dual of $\mathfrak{n}$, and, as we have seen in Chapter 1, the Chevalley-Rosenlicht theorem guarantees the existence of a Zariski-open set $U \subset \mathfrak{n}^*$ of N-coadjoint orbits which are in general position with respect to the basis $\mathcal{B}^*$ (the so-called 'generic' orbits). If $\ell = \sum_{i=1}^n \ell_i X_i^*$ and $\mathcal{O}_\ell = \mathrm{Ad}^*(N)\ell \subset U$, then $\mathcal{O}_\ell$ has maximal dimension, which we shall say is $2k$, where $n - 2k = \dim \mathfrak{r}_\ell = r$.

The Chevalley-Rosenlicht theorem also shows that there exists a set of positive integers $J = \{j_1, \ldots, j_{2k}\}$, $2 \leq j_1 < \ldots < j_{2k} \leq n$, such that every generic orbit $\mathcal{O}_\ell$ lies over the $2k$-dimensional subspace $W_J = \mathbb{R}\text{-span}\{X_{j_1}^*, \ldots, X_{j_{2k}}^*\}$. We call J the set of orbit (or jump) indices determined by the basis $\mathcal{B}^*$. If $D = \{d_1, \ldots, d_r\}$, $1 = d_1 < \ldots < d_r \leq n$, where $J \cup D = \{1, \ldots, n\}$ and $J \cap D = \emptyset$, then the r-dimensional subspace $W_D = \mathbb{R}\text{-span}\{X_{d_1}^*, \ldots, X_{d_r}^*\}$ intersects U in a cross-section of the set of generic orbits and $\widehat{N}_{g.p.} \cong \mathcal{W} = U \cap W_D$, where $\widehat{N}_{g.p.}$ denotes the representations in $\widehat{N}$ in general position. For this reason we call D the set of parameter (or dual) indices. If $\ell \in \mathcal{W}$, we shall write $\ell = \sum_{i=1}^{r} \ell_{d_i} X_{d_i}^*$ and call ℓ a 'generic parametrizing' functional (or 'generic orbit representative'). The components $\ell_{d_1}, \ldots, \ell_{d_r}$ will be called the 'generic parametrizing components' of ℓ (of course these components must satisfy certain conditions, about which more anon). If dm_r^* denotes r-dimensional Lebesgue measure on W_D, then the measure

$$dR = |\text{Pfaffian}(\ell)|\ dm_r^*$$

on the set $\mathcal{W}$ of generic orbit representatives may be transferred to a Borel measure $d\rho$ on $\widehat{N}$ with support $\widehat{N}_{g.p.}$. This is the Plancherel measure for $\widehat{N}$.

§ 2.2 A Structure Theorem for Nilpotent Lie Algebras

We wish to show that there exists in $\mathfrak{n}^*$ a subset $\mathcal{W}_S \subseteq \mathcal{W}$ consisting of what we shall call 'strongly generic parametrizing functionals'. The set $\mathcal{W}_S$ has the following properties:

(1) For each $\ell \in \mathcal{W}_S$, there exists a weak Malcev basis

$$\mathcal{B}_w(\ell) = \{X_1, \ldots, X_c, Y_{m_1}, Y_{m_2}^{\ell}, \ldots, Y_{m_p}^{\ell}, X_{e_1}, \ldots, X_{e_k}\}$$

for $\mathfrak{n}$ such that:

(a) $c + 1 = m_1 < \ldots < m_p \leq n$ and $c + 2 \leq e_1 < \ldots < e_k \leq n$;

(b) The vectors $X_1, \ldots, X_c, Y_{m_1}$ and $X_{e_1}, \ldots, X_{e_k}$ are *fixed* elements of the original strong Malcev basis $\mathcal{B}$ which do not depend on ℓ;

(c) $\mathfrak{z}(\mathfrak{n}) = \mathbb{R}\text{-span}\{X_1, \ldots, X_c\}$ is the center of $\mathfrak{n}$;

(d) $Y_{m_1} = X_{c+1}$;

(e) Each vector $Y^{\ell}_{m_t}$, $2 \le t \le p$, has the form

$$Y^{\ell}_{m_t} = X_{m_t} + \sum_{e_j < m_t} c_{m_t,e_j}(\ell)\, X_{e_j},$$

where X_{m_t} is a fixed element of the basis $\mathcal{B}$ and the coefficients $c_{m_t,e_j}(\ell)$ are rational nonsingular functions which depend only on the parametrizing components $\ell_{d_1}, \ldots, \ell_{d_s}$ of ℓ such that $d_s < m_t$;

(f) $\mathfrak{m}_\ell = \mathbb{R}\text{-span}\{X_1, \ldots, X_c, Y_{m_1}, Y^{\ell}_{m_2}, \ldots, Y^{\ell}_{m_p}\}$ is a polarizing subalgebra for ℓ through which the basis $\mathcal{B}_w(\ell)$ passes (a 'special' polarization).

(2) The r-dimensional Lebesgue measure of $\mathcal{W} \sim \mathcal{W}_S$ is zero, so $\mathcal{W}_S$ contains almost all generic parametrizing functionals, and the set of representations $\pi \in \widehat{N}_{g.p.}$ which are *not* induced from characters χ_ℓ for $\ell \in \mathcal{W}_S$ also has Plancherel measure zero.

In the applications to follow in Chapter 3, we shall employ the standard methods of Mackey [6] and Kirillov [5] to construct a (Schrödinger) model for the representation π_ℓ corresponding to $\ell \in \mathcal{W}_S$. In this model, π_ℓ will act in $L^2(\mathbb{R}\text{-span}\{X_{e_1}, \ldots, X_{e_k}\})$, where L^2 is formed with respect to the ordinary Euclidean measure $dx_1 \cdots dx_k$ in $\mathbb{R}^k$. Thus the Hilbert space $\mathcal{H}_{\pi_\ell}$ for π_ℓ will be *fixed* for all $\ell \in \mathcal{W}_S$. Since we shall wish to decompose Haar measure $d\nu$ on N (Euclidean measure dn on $\mathfrak{n}$) into a Cartesian product of the measure $dx_1 \cdots dx_k$ with a suitable Haar measure $dm(\ell)$ on $M_\ell = \exp\, \mathfrak{m}_\ell$, we shall take $dm(\ell)$ to be the measure on M_ℓ obtained by projection onto the coordinate hyperplane spanned by the set of vectors

$$\mathcal{B} \sim \{X_{e_1}, \ldots, X_{e_k}\} = \{X_1, \ldots, X_c, X_{m_1}, \ldots, X_{m_p}\}$$

from the original strong Malcev basis $\mathcal{B}$. This hyperplane will be equipped with its ordinary $(n-k)$-dimensional Euclidean measure

$$dm_{n-k} = dx_1 \cdots dx_c \, dx_{m_1} \cdots dx_{m_p}.$$

Our choice of a single family of rationally varying (indeed, *rotating*) polarizations for strongly generic representations enables us thereby to fix *one* modelling space shared by *all* strongly generic π_ℓ, and to specify a useful Haar measure $dm(\ell)$ for each M_ℓ corresponding to π_ℓ.

§ 2.3 Theorems of Kirillov and of Vergne

As we shall see, a single method of construction suffices to determine the set $\mathcal{W}_S$, the special polarizations $\mathfrak{m}_\ell$, and the special bases $\mathcal{B}_w(\ell)$. This method rests squarely upon the shoulders of the following two theorems (see [3], pp. 29–30):

2.3.1 Theorem (Kirillov). *Let* $\mathfrak{n}_0$ *be a codimension 1 subalgebra in a nilpotent Lie algebra* $\mathfrak{n}$. *Let* $\ell \in \mathfrak{n}^*$ *and* $\ell^0 = \ell|_{\mathfrak{n}_0}$. *Then if* $\mathfrak{r}_\ell$ *denotes the radical of* ℓ, *there are two mutually exclusive possibilities.*

Case 1, *characterized by any of the following equivalent properties:*

(i) $\mathfrak{r}_\ell \subset \mathfrak{n}_0$

(ii) $\mathfrak{r}_\ell \subset \mathfrak{r}_{\ell^0}$

(iii) $\mathfrak{r}_\ell$ *has codimension 1 in* $\mathfrak{r}_{\ell^0}$. *In this case, any subalgebra which polarizes* ℓ^0 *also polarizes* ℓ.

Case 2, *characterized by any of the following equivalent properties:*

(i) $\mathfrak{r}_\ell \not\subset \mathfrak{n}_0$

(ii) $\mathfrak{r}_{\ell^0} \subset \mathfrak{r}_\ell$

(iii) $\mathfrak{r}_{\ell^0}$ *has codimension 1 in* $\mathfrak{r}_\ell$. *In this case, if* $\mathfrak{m}$ *is a polarizing subalgebra for* ℓ, *then* $\mathfrak{m}_0 = \mathfrak{m} \cap \mathfrak{n}_0$ *is a polarizing subalgebra for* ℓ^0; *also,* $\mathfrak{m}_0$ *has codimension 1 in* $\mathfrak{m}$ *and* $\mathfrak{m} = \mathfrak{m}_0 + \mathfrak{r}_\ell$.

2.3.2 Theorem (Vergne). *Let* $\mathcal{B} = \{X_1, \ldots, X_n\}$ *be a strong Malcev basis for a nilpotent Lie algebra* $\mathfrak{n}$, *and let* $\mathfrak{n}_j = \mathbb{R}\text{-span}\{X_1, \ldots, X_j\}$. *Let* $\ell \in \mathfrak{n}^*$ *and let* $\ell^j = \ell|_{\mathfrak{n}_j}$. *Then*

$$\mathfrak{m}_\ell = \sum_{j=1}^{n} \mathfrak{r}_{\ell^j}$$

is a polarizing subalgebra for ℓ, *where*

$$\mathfrak{r}_{\ell^j} = \{X \in \mathfrak{n}_j : \ \ \mathrm{ad}^*(X)\ell^j = 0\}$$

is the radical of ℓ^j.

We shall use these two theorems as follows. Suppose that $\ell^j = \ell|_{\mathfrak{n}_j}$ is the restriction to $\mathfrak{n}_j$ of a generic functional in U, and suppose also that $\ell^j \in W_D$, the set of parametrizing functionals. Then if

$$\mathfrak{m}_{\ell^{j-1}} = \sum_{i=1}^{j-1} \mathfrak{r}_{\ell^i}$$

has already been contructed, Theorem 2.3.1 tells us that

(1) either $\mathfrak{m}_{\ell j} = \mathfrak{m}_{\ell j-1}$, which is the case if $\mathfrak{r}_{\ell j} \subset \mathfrak{r}_{\ell j-1}$;

(2) or else $\mathfrak{m}_{\ell j} = \mathfrak{m}_{\ell j-1} + \mathfrak{r}_{\ell j}$, which is the case if $\mathfrak{r}_{\ell j-1} \subset \mathfrak{r}_{\ell j}$.

Since $\mathfrak{m}_{\ell 1} = \mathfrak{r}_{\ell 1} = \mathbb{R} X_1$ always, it is clear that our task at the j-th stage of the construction of $\mathfrak{m}_\ell$ consists, at worst, in finding a vector Y_j such that

$$\mathfrak{r}_{\ell j} = \mathfrak{r}_{\ell j-1} + \mathbb{R} Y_j \ .$$

For then

$$\begin{aligned} \mathfrak{m}_{\ell j} &= \mathfrak{m}_{\ell j-1} + \mathfrak{r}_{\ell j} \\ &= \mathfrak{m}_{\ell j-1} + \mathfrak{r}_{\ell j-1} + \mathbb{R} Y_j \\ &= \mathfrak{m}_{\ell j-1} + \mathbb{R} Y_j \,, \end{aligned}$$

since $\mathfrak{r}_{\ell j-1} \subseteq \mathfrak{m}_{\ell j-1}$. After n stages, Theorem 2.3.2 tells us that $\mathfrak{m}_{\ell^n} = \mathfrak{m}_\ell$ is a polarizing subalgebra for ℓ.

§ 2.4 Some Matrix Notation

Since the strong Malcev basis $\mathcal{B}$ for $\mathfrak{n}$ has been fixed for the duration (as is necessary for any definition of orbits in general position), we shall work with *matrix* versions of several objects. We take a moment to fix some notation. Recall that $B_\ell : \mathfrak{n} \times \mathfrak{n} \to \mathbb{R}$ denotes the antisymmetric bilinear form given by

$$B_\ell(W_1, W_2) = \ell\big([W_1, W_2]\big) \qquad \text{for } \ell \in \mathfrak{n}^* \text{ and } W_1, W_2 \in \mathfrak{n} .$$

Let $[B_\ell]$ denote the matrix of B_ℓ with respect to the basis $\mathcal{B}$, and let $[B_\ell]_{j\times j}$ denote the upper left-hand $(j \times j)$-block of $[B_\ell]$. Let $[X]$ denote the vector $X \in \mathfrak{n}$ written as a column vector with respect to $\mathcal{B}$ (so $[X] \in \mathbb{R}^n$), and let $[X]_{j\times 1}$ denote the column vector consisting of the top j entries of $[X]$ (so $[X]_{j\times 1} \in \mathbb{R}^j$). Recall that the radical of $\ell^j = \ell|_{\mathfrak{n}_j}$ is

$$\begin{aligned}\mathfrak{r}_{\ell j} &= \{X \in \mathfrak{n}_j : \ \ell^j([X,Y]) = 0 \ \text{ for all } \ Y \in \mathfrak{n}_j\} \\ &= \{X \in \mathfrak{n}_j : \ \mathrm{ad}^*(X)\ell^j = 0\}.\end{aligned}$$

In matrix notation we may write, for $X \in \mathfrak{n}_j$,

$$\mathrm{ad}^*(X)\ell^j = [B_\ell]_{j\times j}\ [X]_{j\times 1}\,,$$

where juxtaposition signifies matrix multiplication. So

$$\mathfrak{r}_{\ell j} \cong \big\{\, [X]_{j\times 1} \in \mathbb{R}^j : \ [B_\ell]_{j\times j}\ [X]_{j\times 1} = 0 \big\}.$$

§ 2.5 Outline of the Construction Method

Suppose now that we have constructed $\mathfrak{m}_{\ell j-1}$ and are seeking to construct $\mathfrak{m}_{\ell j}$. We know that if $\mathfrak{r}_{\ell j} \subset \mathfrak{r}_{\ell j-1}$ we may simply set $\mathfrak{m}_{\ell j} = \mathfrak{m}_{\ell j-1}$, while if $\mathfrak{r}_{\ell j-1} \subset \mathfrak{r}_{\ell j}$ we may set $\mathfrak{m}_{\ell j} = \mathfrak{m}_{\ell j-1} + \mathbb{R}Y_j$. Thus for $1 \leq j \leq n$ we first need a way of deciding whether $\mathfrak{r}_{\ell j} \subset \mathfrak{r}_{\ell j-1}$ or $\mathfrak{r}_{\ell j-1} \subset \mathfrak{r}_{\ell j}$, and then, if the latter inclusion holds, we need a way of finding a vector Y_j with the requisite property (of course, Y_j will not be unique and will, in general, depend on ℓ).

Consider the linear system

$$(*) \qquad [B_\ell]_{j\times j}\ [X]_{j\times 1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}_{j\times 1} .$$

As we have just seen, the set of all vectors $[X]_{j\times 1}$ satisfying this system is (isomorphic to) $\mathfrak{r}_{\ell j}$. And as we shall see in § 2.9, there exists a well-defined procedure for row- and column-reducing the matrix $[B_\ell]_{j\times j}$ which always produces either an antisymmetric matrix with a bottom (j-th) row of zeros, or else an antisymmetric matrix with a single non-zero entry on its bottom row, which entry is also the only non-zero entry in its column.

Suppose, on the one hand, that the bottom row of the *reduced* matrix $[B_\ell]_{j\times j}$ consists of zeros. Then, by antisymmetry, the right-hand (j-th) column also consists of zeros, from which it follows that any solution to $(*)$ is unconstrained in the variable x_j. Hence $\mathfrak{r}_{\ell j} \not\subset \mathfrak{n}_{j-1} = \mathbb{R}\text{-span}\{X_1, \ldots, X_{j-1}\}$, and Case 2 of Theorem 2.3.1 implies that $\mathfrak{r}_{\ell j-1} \subset \mathfrak{r}_{\ell j}$. On the other hand, suppose that the bottom row of the *reduced* matrix $[B_\ell]_{j\times j}$ contains a single non-zero entry on its bottom row, which entry has only zeros above it. Then, by antisymmetry, the right-hand column contains only a single non-zero entry, which entry has only zeros to its left, and it follows that any solution to $(*)$ must satisfy $x_j = 0$. Hence $\mathfrak{r}_{\ell j} \subset \mathfrak{n}_{j-1}$, and Case 1 of Theorem 2.3.1 implies that $\mathfrak{r}_{\ell j} \subset \mathfrak{r}_{\ell j-1}$.

At the j-th stage of our construction process, then, it appears that we must accomplish three things:

(1) insure that ℓ^j is the restriction to $\mathfrak{n}_j$ of a linear functional $\ell \in \mathcal{W}$, where $\mathcal{W} = U \cap W_D$;

(2) insure that $[B_\ell]_{j\times j}$ is fully row- and column-reduced (in a manner to be specified below);

(3) construct $\mathfrak{m}_{\ell j}$, which may involve finding an (in general) ℓ-dependent vector Y_j which satisfies the equation $\mathfrak{m}_{\ell j} = \mathfrak{m}_{\ell j-1} + \mathbb{R} Y_j$.

These three tasks may in fact be carried out simultaneously, as we now proceed to show. Keep in mind that our aim is to produce the set $\mathcal{W}_S$ and, for each

$\ell \in \mathcal{W}_S$, a special polarization $\mathfrak{m}_\ell$ and a special weak Malcev basis $\mathcal{B}_w(\ell)$ which passes through $\mathfrak{m}_\ell$.

§ 2.6 Illustration of the Construction Method

We may at our pleasure fix a strong Malcev basis $\mathcal{B}$ for $\mathfrak{n}$ which passes through the center $\mathfrak{z} = \mathbb{R}\text{-span}\{X_1, \ldots, X_c\}$. Since $B_\ell(X_i, X_j) = 0$ for $1 \leq i, j \leq c+1$, the top c rows and the left-hand c columns of $[B_\ell]$ consist of zeros, and so it is obvious that the upper left-hand $(c+1) \times (c+1)$ block of $[B_\ell]$ is the $(c+1) \times (c+1)$ zero matrix for all $\ell \in \mathfrak{n}^*$. Indeed,

$$[B_\ell] = \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & b_{c+1,c+2} & \cdots & \cdots & \cdots & b_{c+1,n} \\ 0 & \cdots & 0 & b_{c+2,c+1} & 0 & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & 0 & b_{n-1,n} \\ 0 & \cdots & 0 & b_{n,c+1} & \cdots & \cdots & \cdots & b_{n,n-1} & 0 \end{bmatrix}_{n \times n},$$

where the entry $b_{i,j}$ is, in general, a polynomial in the first $i-1$ components of ℓ. Hence $\mathfrak{r}_{\ell^{c+1}} = \mathbb{R}\text{-span}\{X_1, \ldots, X_{c+1}\}$, and we may always set $\mathfrak{m}_{\ell^{c+1}} = \mathfrak{r}_{\ell^{c+1}}$. Setting $Y_{m_1} = X_{c+1}$, we have the first $c+1$ vectors of all of the bases $\mathcal{B}_w(\ell)$ that we are seeking to construct. We note for the record that Y_{m_1} is the first "internal orbit vector"† in the bases and we set $i_1 = m_1$ to signify this fact. (It will always be the case that $i_1 = m_1 = c+1$ and that the first "jump" index is $j_1 = i_1$.) In addition, we are free to set $\ell_{i_1} = 0$ since components in orbit directions are not parametrizing components (there will be $2k$ parametrization conditions before we are done). By the way, for obvious reasons we shall henceforward gently abuse notation by regarding $[B_\ell]$ as the $(n-c)\times(n-c)$ matrix obtained by deleting the top rows and and left-hand columns of zeros. Thus, we shall write (usually without comment)

$$[B_\ell] = \begin{bmatrix} 0 & b_{c+1,c+2} & \cdots & \cdots & \cdots & b_{c+1,n} \\ b_{c+2,c+1} & 0 & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & 0 & b_{n-1,n} \\ b_{n,c+1} & \cdots & \cdots & \cdots & b_{n,n-1} & 0 \end{bmatrix}_{n-c\ \times\ n-c},$$

or, remembering that B_ℓ is an antisymmetric bilinear form,

† As we have already mentioned, generic N-coadjoint orbits lie suspended over the subspace $W_D = \mathbb{R}\text{-span}\{X^*_{j_1}, \ldots, X^*_{j_{2k}}\}$, where $J = \{j_1 < \cdots < j_{2k}\}$ is the set of orbit indices. We are calling a vector in the basis $\mathcal{B}$ an "internal orbit vector" if it is indexed by some $j_i \in J$ and if it is also in $\mathfrak{m}_\ell$. Similarly, a vector in $\mathcal{B}$ will be called an "external orbit vector" if it is indexed by some $j_i \in J$ and if it lies outside $\mathfrak{m}_\ell$. We let $J_I = \{i_1, \ldots, i_k\}$ and $J_E = \{e_1, \ldots, e_k\}$ denote, respectively, the sets of internal and external orbit indices.

$$[B_\ell] = \begin{bmatrix} 0 & b_{c+1,c+2} & \cdots & \cdots & \cdots & b_{c+1,n} \\ -b_{c+1,c+2} & 0 & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & 0 & b_{n-1,n} \\ -b_{c+1,n} & \cdots & \cdots & \cdots & -b_{n-1,n} & 0 \end{bmatrix}_{n-c \,\times\, n-c} .$$

Suppose, for illustrative purposes, that the entry $b_{c+1,c+2}$ is not identically zero (and so is a non-trivial polynomial in the components $\ell_1, \ldots, \ell_c$ of ℓ). Then there are two alternatives:

(1) we set $b_{c+1,c+2} = 0$, in which case

$$\mathfrak{r}_{\ell\, c+2} = \mathfrak{r}_{\ell\, c+1} + \mathbb{R} X_{c+2} \supset \mathfrak{r}_{\ell\, c+1},$$

and we may proceed as in the $c+1$ case;

(2) we set $b_{c+1,c+2} \neq 0$ (that is, we disregard functionals ℓ for which $b_{c+1,c+2} = 0$), in which case

$$\mathfrak{r}_{\ell\, c+2} = \mathfrak{r}_{\ell\, c} \subset \mathfrak{r}_{\ell\, c+1},$$

and we may set $\mathfrak{m}_{\ell\, c+2} = \mathfrak{m}_{\ell\, c+1}$, then set $X_{e_1} = X_{c+2}$ (the first "external orbit vector"), and then move on.

The problem with alternative (1) is that it can prevent a functional from being generic. To see this, recall from Chapter 1 that a linear functional $\ell \in \mathfrak{n}^*$ is called 'generic' if, speaking in matrix terms now, the rank of $[B_\ell]_{j \times n}$ is as large as possible for each $1 \leq j \leq n$ (see also [3], p.86). Equivalently, ℓ is generic if the dimension of $\mathfrak{a}_j$, the annihilator of $[B_\ell]_{j \times n}$, is as small as possible for each $j = 1, \ldots, n$, where

$$\begin{aligned}\mathfrak{a}_j &= \{X \in \mathfrak{n} \,:\, \mathrm{ad}^*(X)\ell^j = 0\} \\ &= \{\, [X] \in \mathbb{R}^n : \, [B_\ell]_{j\times n}\, [X] = 0\}.\end{aligned}$$

Insuring the genericity of a functional ℓ requires us (in general) to *prohibit* certain polynomials in the components of ℓ from being zero, not *set* them equal to zero. In short, alternative (1) is sure to get us into trouble eventually.†

It would seem, then, that we are encouraged by the facts of the matter to embrace alternative (2). If $b_{c+1,c+2}$ is not identically zero and we then prohibit it from being zero, we do in fact make the ranks of the rectangular matrices $[B_\ell]_{(c+1)\times n}$ and $[B_\ell]_{(c+2)\times n}$ as large as possible (that is, 1 and 2, respectively). So we are on our way to genericity for ℓ. And we see that $i_1 = c+1$ and $e_1 = c+2$ are internal and external orbit indices, respectively, and we may set $\ell_{i_1} = \ell_{e_1} = 0$, since, again, components in orbit directions are not parametrizing components.

But what do we do about $\mathfrak{r}_{\ell}{}^{c+3}$? Now we are confronted with the matrix

$$[B_\ell]_{(c+3)\times(c+3)} = \begin{bmatrix} 0 & b_{c+1,c+2} & b_{c+1,c+3} \\ -b_{c+1,c+2} & 0 & b_{c+2,c+3} \\ -b_{c+1,c+3} & -b_{c+2,c+3} & 0 \end{bmatrix}.$$

† If an example is wanted, one need look only as far as the 3-dimensional Heisenberg group H_1. If its Lie algebra $\mathfrak{h}_1$ is viewed as $\mathbb{R}$-span$\{X_1, X_2, X_3\}$, where $[X_3, X_2] = X_1$, then for $\ell \in \mathfrak{h}_1^*$ we have

$$[B_\ell] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \ell_1 \\ 0 & -\ell_1 & 0 \end{bmatrix}.$$

Setting $\ell_1 = 0$ forces the rank of $[B_\ell]$ to be 0, whereas setting $\ell_1 \neq 0$ forces the rank of $[B_\ell]$ to be 2. Of course, for $\mathfrak{h}_1$ the set U of generic functionals consists precisely of those ℓ such that $\ell_1 \neq 0$.

We must decide whether $\mathfrak{r}_{\ell\, c+2} \subset \mathfrak{r}_{\ell\, c+3}$ or $\mathfrak{r}_{\ell\, c+3} \subset \mathfrak{r}_{\ell\, c+2}$. And if the former inclusion holds, we must find a vector Y_{c+3} such that

$$\mathfrak{m}_{\ell\, c+3} = \mathfrak{m}_{\ell\, c+2} + \mathbb{R}\, Y_{c+3}.$$

Recall that the solution set of the linear system

$$(*) \qquad [B_\ell]_{(c+3)\times(c+3)}\,[X]_{(c+3)\times 1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}_{(c+3)\times 1}$$

is algebraically isomorphic to $\mathfrak{r}_{\ell\, c+3}$. To solve the system, it suffices to row-reduce $[B_\ell]_{(c+3)\times(c+3)}$. Under the continuing temporary hypothesis that $b_{c+1,c+2} \neq 0$, we find that this matrix reduces to

$$\begin{bmatrix} 0 & b_{c+1,c+2} & b_{c+1,c+3} \\ -b_{c+1,c+2} & 0 & b_{c+2,c+3} \\ 0 & 0 & 0 \end{bmatrix},$$

where we have

(1) added $(b_{c+2,c+3} \times \text{ row } 1)$ to $(b_{c+1,c+2} \times \text{ row } 3)$, and

(2) added $(-b_{c+1,c+3} \times \text{ row } 2)$ to $(b_{c+1,c+2} \times \text{ row } 3)$.

Now since $[B_\ell]$ and $[B_\ell]_{(c+3)\times(c+3)}$ are antisymmetric matrices, after row-reducing we should also column-reduce to restore the antisymmetry. If we do this now to the above row-reduced version of $[B_\ell]_{(c+3)\times(c+3)}$, we get the antisymmetric matrix

$$\begin{bmatrix} 0 & b_{c+1,c+2} & 0 \\ -b_{c+1,c+2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The right-hand column of zeros tells us that any solution to the system $(*)$ is unconstrained in the variable x_{c+3}, whence $\mathfrak{r}_{\ell\,c+3} \not\subset \mathfrak{n}_{c+2}$. It then follows from Case 2 of Theorem 2.3.1 that $\mathfrak{r}_{\ell\,c+2} \subset \mathfrak{r}_{\ell\,c+3}$, and so

$$\mathfrak{m}_{\ell\,c+3} = \mathfrak{m}_{\ell\,c+2} + \mathbb{R}\,Y_{c+3}$$

for some ℓ-dependent vector Y_{c+3}.

§ 2.7 Lessons of the Illustration

A few comments are in order here. In the first place, we row-reduced as if the entries of $[B_\ell]$ were scalars in a ring (rather than a field) for reasons of typographical convenience. In the second place, since $[B_\ell]$ and $[B_\ell]_{(c+3)\times(c+3)}$ are antisymmetric matrices, after row-reducing we also column-reduced to restore the antisymmetry. In the third place, we did not interchange rows in the reduction process (in an attempt to produce a so-called "echelon form"). The reason for this is simple: $\mathfrak{n}$ is more than just a vector space—it is an algebra. Indeed, it is an algebra in which a strong Malcev basis has been fixed. To interchange two rows in $[B_\ell]$ is to interchange two vectors in $\mathcal{B}$, and so to run the risk of losing the 'strength' of the basis, which consists in the fact that $\mathbb{R}$-span$\{X_1,\ldots,X_j\}$ is an ideal for $1 \leq j \leq n$. Since that strength will be used later on (in § 2.10, to be exact), row interchanges must be forbidden. In the fourth place, the reduction of $[B_\ell]_{(c+3)\times(c+3)}$ employed the antisymmetric pair $b_{c+1,c+2}$ and $-b_{c+1,c+2}$, and relied upon the hypothesis that $b_{c+1,c+2} \neq 0$. It is clear that to determine the containment relations of $\mathfrak{r}_{\ell\, c+2}$ and $\mathfrak{r}_{\ell\, c+3}$ only one reduction by one antisymmetric pair was necessary. This phenomenon is general: to determine the containment relations of $\mathfrak{r}_{\ell\, j}$ and $\mathfrak{r}_{\ell\, j+1}$, at most one reduction by one antisymmetric pair is necessary (of course, *no* reduction may be necessary or even possible). Each such reduction will require us to prohibit a non-trivial polynomial in (some of) the components of ℓ from being zero (these prohibitions will be called 'strong genericity conditions', for reasons to be explained). So, summing up the lessons of this special case, we shall in the future always

(1) reduce as if working over a ring rather than a field;

(2) antisymmetrize after each reduction;

(3) refrain from interchanging rows during reduction;

(4) reduce (when necessary) by using antisymmetric pairs which have been subjected to a strong genericity condition, drawing conclusions as each pair is used.

§ 2.8 A Less Than Rare Device

There is still another problem that we need to address before discussing the general case: how do we find the vector Y_{c+3}? In fact, we need a device to produce this vector for us, and we find this device in a special kind of augmented matrix. Let

$$[A_\ell]_{(c+3)\times(c+4)} = \left[\begin{array}{c|c} & X_{c+1} \\ [B_\ell]_{(c+3)\times(c+3)} & X_{c+2} \\ & X_{c+3} \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 0 & b_{c+1,c+2} & b_{c+1,c+3} & X_{c+1} \\ -b_{c+1,c+2} & 0 & b_{c+2,c+3} & X_{c+2} \\ -b_{c+1,c+3} & -b_{c+2,c+3} & 0 & X_{c+3} \end{array}\right],$$

where the entries in the right-hand column—the 'bookkeeping' column, as we shall call it—are not scalars, but rather vectors from $\mathcal{B}$, the fixed strong Malcev basis for $\mathfrak{n}$. If we now row-reduce and antisymmetrize $[B_\ell]_{(c+3)\times(c+3)}$ as before, the bookkeeping column records the ℓ-dependent basis changes wrought thereby. Indeed, the following matrix results:

$$\left[\begin{array}{ccc|c} 0 & b_{c+1,c+2} & 0 & X_{c+1} \\ -b_{c+1,c+2} & 0 & 0 & X_{c+2} \\ 0 & 0 & 0 & \widetilde{X}_{c+3} \end{array}\right],$$

where

$$\widetilde{X}_{c+3} = b_{c+1,c+2}X_{c+3} - b_{c+1,c+3}X_{c+2} + b_{c+2,c+3}X_{c+1}.$$

This vector has the property that $\mathfrak{r}_{\ell\, c+3} = \mathfrak{r}_{\ell\, c+2} + \widetilde{X}_{c+3}$, where $\mathfrak{r}_{\ell\, c+2}$ is, of course, just the center $\mathfrak{z}$ of $\mathfrak{n}$. Hence, $\mathfrak{m}_{\ell\, c+3} = \mathfrak{m}_{\ell\, c+2} + \widetilde{X}_{c+3}$, and we see that this vector is a fine choice for the vector Y_{c+3} that we seek. However, we can do better than this. Since $X_{c+1} \in \mathfrak{m}_{\ell\, c+2}$ (as seen above), we may set

$$\mathfrak{m}_{\ell\, c+3} = \mathfrak{m}_{\ell\, c+2} + \widetilde{\widetilde{X}}_{c+3},$$

where

$$\widetilde{\widetilde{X}}_{c+3} = b_{c+1,c+2}X_{c+3} - b_{c+1,c+3}X_{c+2}.$$

Now, $i_2 = c+3$ is the second internal orbit index, and $\widetilde{\widetilde{X}}_{c+3}$ depends only upon ℓ, X_{i_2}, and $X_{e_1} = X_{c+2}$, the first *external* orbit vector. If we set $m_2 = i_2$, we may write

$$\begin{aligned} Y^\ell_{m_2} &= \frac{1}{b_{c+1,c+2}}\widetilde{\widetilde{X}}_{c+3} \\ &= X_{m_2} + c_{m_2,e_1}(\ell)X_{e_1}, \end{aligned}$$

where

$$c_{m_2,e_1}(\ell) = -\frac{b_{c+1,c+3}}{b_{c+1,c+2}}$$

is a rational nonsingular function of those $\ell \in \mathfrak{n}^*$ satisfying the strong genericity condition $b_{c+1,c+2} \neq 0$ and the parametrization conditions $\ell_{i_1} = \ell_{e_1} = 0$. The vector $Y^{\ell}_{m_2}$ is perhaps the best choice for a vector which spans $\mathfrak{m}_{\ell\,c+3} \sim \mathfrak{m}_{\ell\,c+2}$. If we compare this vector to item (1e) in §2.2 above, we see that its form is exactly as advertised.

The vector $\widetilde{X}_{m_2} = \widetilde{X}_{c+3}$ spans that part of the radical $\mathfrak{r}_{\ell\,c+3}$ which is not already contained in $\mathfrak{r}_{\ell\,c+2}$ (i.e., $\mathfrak{r}_{\ell\,c+3} \sim \mathfrak{r}_{\ell\,c+2}$), while the vector $\widetilde{\widetilde{X}}_{m_2} = \widetilde{\widetilde{X}}_{c+3}$ spans that part of the radical $\mathfrak{r}_{\ell\,c+3}$ which is not already contained in $\mathfrak{m}_{\ell\,c+2}$ (i.e., $\mathfrak{m}_{\ell\,c+3} \sim \mathfrak{m}_{\ell\,c+2}$). Since it is only the latter vector which interests us, let us adapt our augmented matrix device to produce such a vector. Indeed, all we need do is stipulate that once a bookkeeping column entry has been found to be an element of some $\mathfrak{m}_{\ell j}$, that entry is immediately replaced by zero in the bookkeeping column. This has the effect of eliminating linear redundancies among the basis elements of $\mathfrak{m}_\ell$. In the present example, the vector X_{c+1} should have been replaced by zero as soon as it was seen to be an element of $\mathfrak{m}_{\ell\,c+1}$ (and so of $\mathfrak{m}_\ell$). That is, instead of looking like this:

$$\left[\begin{array}{ccc|c} 0 & b_{c+1,c+2} & b_{c+1,c+3} & X_{c+1} \\ -b_{c+1,c+2} & 0 & b_{c+2,c+3} & X_{c+2} \\ -b_{c+1,c+3} & -b_{c+2,c+3} & 0 & X_{c+3} \end{array}\right],$$

the augmented matrix $[A_\ell]_{(c+3)\times(c+4)}$ should have looked like *this*:

$$\left[\begin{array}{ccc|c} 0 & b_{c+1,c+2} & b_{c+1,c+3} & 0 \\ -b_{c+1,c+2} & 0 & b_{c+2,c+3} & X_{c+2} \\ -b_{c+1,c+3} & -b_{c+2,c+3} & 0 & X_{c+3} \end{array}\right].$$

Then, reducing as we did before, we would have arrived at the matrix

$$\left[\begin{array}{ccc|c} 0 & b_{c+1,c+2} & 0 & 0 \\ -b_{c+1,c+2} & 0 & 0 & X_{c+2} \\ 0 & 0 & 0 & {}^{1}X_{c+3} \end{array}\right],$$

where we have replaced $\tilde{\tilde{X}}_{c+3}$ by ${}^{1}X_{c+3}$ (the pre-superscript notation is obviously preferable to the tilde-stacking notation for showing how many reductions have been performed). But remember: ${}^{1}X_{c+3}$ does *not* span $\mathfrak{r}_{\ell^{c+3}} \sim \mathfrak{r}_{\ell^{c+2}}$, but only $\mathfrak{m}_{\ell^{c+3}} \sim \mathfrak{m}_{\ell^{c+2}}$. Fortunately, this is enough for our purposes.

§ 2.9 The General Construction

We are ready now to generalize the concepts and techniques developed over the last few pages under the temporary hypothesis that $b_{c+1,c+2} \neq 0$. Suppose that $\ell \in \mathfrak{n}^*$ is perfectly arbitrary. Since $\mathfrak{z}(\mathfrak{n}) = \mathbb{R}\text{-span}\{X_1, \ldots, X_c\}$, we have $\mathfrak{r}_{\ell^c} = \mathfrak{z}(\mathfrak{n})$ and we may set $\mathfrak{m}_{\ell^c} = \mathfrak{r}_{\ell^c} = \mathfrak{z}(\mathfrak{n})$. So our work begins at row $c+1$ of $[B_\ell]$, or rather at row $c+1$ of the augmented matrix

$$[A_\ell] = \left[\begin{array}{cccccc|c} 0 & b_{c+1,c+2} & \cdots\ \cdots & \cdots & b_{c+1,n} & & X_{c+1} \\ -b_{c+1,c+2} & 0 & \cdots\ \cdots & \cdots & \cdots & & X_{c+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots\ \cdots & 0 & b_{n-1,n} & & X_{n-1} \\ -b_{c+1,n} & \cdots & \cdots\ \cdots & -b_{n-1,n} & 0 & & X_n \end{array}\right].$$

Note that $[A_\ell]$ has not as yet undergone any reductions by antisymmetric pairs. This will soon change.

Now because the basis vector $X_{c+1} \in \mathcal{B}$ is non-central, row $c+1$ of $[A_\ell]$ contains at least one entry which is a non-trivial polynomial in the components

$\ell_1, \ldots, \ell_c$ of $\ell^{c+1} = \ell|_{\mathfrak{n}_{c+1}}$ (this entry need not be $b_{c+1,c+2}$, as we assumed above). Let L_{c+1} denote the column in which the *leftmost* such entry is to be found, and let $b_{c+1,L_{c+1}}$ denote the entry itself. Restrict ℓ by prohibiting $b_{c+1,L_{c+1}}$ from being zero; that is, set

$$(SGC\ 1) \qquad\qquad b_{c+1,L_{c+1}} \neq 0.$$

This is the first of our strong genericity conditions (note that it insures that the rank of $[B_\ell]_{(c+1)\times n}$ is 1, which is as large as possible). We use the adjective 'strong' to point up the possibility that there exist generic functionals in $\mathfrak{n}^*$ for which $b_{c+1,L_{c+1}} = 0$, and so we are beginning here to pick out a subset of the full set of generic functionals in $\mathfrak{n}^*$.†

Since row $c+1$ contains a non-zero entry, $c+1 \in J$, the set of orbit indices. Also, $L_{c+1} \in J$ since $b_{L_{c+1},c+1} \neq 0$. In fact, $c+1 = i_1$ is the first internal orbit index, while L_{c+1} is an external orbit index (though we cannot as yet say which one it is). We restrict ℓ further by requiring that

$$(PC\ 1) \qquad\qquad \ell_{c+1} = \ell_{L_{c+1}} = 0.$$

This is our first parametrization condition. The two conditions $(SGC\ 1)$ and $(PC\ 1)$ together insure that ℓ^{i_1} is, as desired, the restriction to $\mathfrak{n}_{c+1}$ of a functional $\ell \in \mathcal{W} = U \cap W_D$

Next we note that $L_{c+1} > c+1$, so the entry $b_{c+1,L_{c+1}}$ is above the diagonal of $[A_\ell]$ (of course, we are abusing language here just a bit since $[A_\ell]$ is not a

† The algebra labelled $\mathfrak{g}_{6,17}$ in Nielsen ([7], p.84) provides an example in which the set of strongly generic parametrizing functionals is a proper subset of the set of generic parametrizing functionals. The non-zero brackets are generated by $[X_6, X_5] = X_1, [X_6, X_3] = X_2, [X_5, X_4] = X_2, [X_4, X_3] = X_1$. It is easy to see that the SGPF's satisfy the condition $\ell_1(\ell_1^2 + \ell_2^2) \neq 0$, while the GPF's satisfy the weaker condition $(\ell_1^2 + \ell_2^2) \neq 0$.

square matrix). Thus row $c+1$ consists of zeros out to column $c+1$, indeed out to column $(L_{c+1} - 1)$. This means, as we know, that $\mathfrak{r}_{\ell c} \subset \mathfrak{r}_{\ell c+1}$, and so we may set $\mathfrak{m}_{\ell c+1} = \mathfrak{m}_{\ell c} + \mathbb{R} X_{c+1}$, where X_{c+1} is the bookkeeping column entry on row $c+1$. It is worth remarking that the entries on row $c+1$ between column L_{c+1} and the bookkeeping column have no bearing on the question whether or not X_{c+1} spans $\mathfrak{m}_{\ell c+1} \sim \mathfrak{m}_{\ell c}$. This is because $\mathcal{B}^* = \{X_1^*, \ldots, X_n^*\}$ is a Jordan-Hölder basis for $\mathfrak{n}^*$. Finally, we set $Y_{m_1} = X_{c+1}$ and replace X_{c+1} by zero in the bookkeeping column.

We are ready now to row-reduce and antisymmetrize $[A_\ell]$ (rather than an upper left-hand block thereof). Since $b_{c+1,L_{c+1}} \neq 0$ and since it is both the leftmost non-zero entry on row c+1 and the topmost non-zero entry in column L_{c+1}, we may use it to sweep out column L_{c+1}. If $c+2 \leq j \leq n$, and $j \neq L_{c+1}$,

(1) we multiply row j of $[A_\ell]$ by $b_{c+1,L_{c+1}}$;

(2) then we add $(-b_{j,L_{c+1}} \times \text{row } c+1)$ to row j, producing a new row j whose (j, L_{c+1})-entry is zero.

Now $b_{c+1,L_{c+1}}$ is the only non-zero entry in column L_{c+1}. (Of course, if it already was the only such entry, then no row operations were necessary.)

By the antisymmetry of $[B_\ell]$, we have that $b_{L_{c+1},c+1} = -b_{c+1,L_{c+1}}$, and so $b_{L_{c+1},c+1}$ is both the topmost non-zero entry in column $c+1$ and the leftmost non-zero entry on row L_{c+1}. Thus we may use $b_{L_{c+1},c+1}$ to sweep out column $c+1$. For each $L_{c+1} < j \leq n$, add $(b_{j,c+1} \times \text{row } L_{c+1})$ to row j, producing a new row j whose $(j, c+1)$-entry is zero. Now $b_{L_{c+1},c+1}$ is the only non-zero entry in column $c+1$. (Again, if it already was the only such entry, then no row operations were necessary).

Having row-reduced $[A_\ell]$ (perhaps vacuously) using $b_{c+1,L_{c+1}}$ and its antisymmetric partner $b_{L_{c+1},c+1}$, we now column-reduce by the same pair. We call the resulting antisymmetrized matrix ${}^1[A_\ell]$ (antisymmetrized modulo the bookkeeping column, of course). This matrix has the form

$$
{}^1[A_\ell] = \left[\begin{array}{c|c} & 0 \\ & {}^1X_{c+2} \\ & \vdots \\ {}^1[B_\ell] & {}^1X_j \\ & \vdots \\ & {}^1X_{n-1} \\ & {}^1X_n \end{array} \right],
$$

where for $c+2 \le j \le n$,

$$
{}^1X_j = \begin{cases} X_j, & \text{if } j = L_{c+1}; \\ b_{c+1,L_{c+1}}\, X_j + b_{j,c+1}\, X_{L_{c+1}} & \text{otherwise.} \end{cases}
$$

In this expression for ${}^1X_j$, the coefficient $b_{j,c+1}$ may well be zero, but the coefficient $b_{c+1,L_{c+1}}$ is *never zero* because of (SGC 1). As a consequence, the vectors ${}^1X_j$ always have a non-zero component in the direction of X_j, and further reduction of ${}^1[B_\ell]$ will not alter this fact. Indeed, if s reductions have been performed on $[A_\ell]$, producing the matrix ${}^s[A_\ell]$, then the bookkeeping column entry ${}^sX_j$, if it has not already been zeroed out, will always have a non-zero component in the direction of X_j (because the coefficient of X_j in ${}^sX_j$ will be a product of polynomials in the parametrizing components of ℓ, each factor of which has been restricted to be non-zero by a strong genericity condition). For this reason, we may speak of ${}^sX_j$ as being an "orbit" vector if $j \in J$, the set of orbit indices for the basis $\mathcal{B}$. By this we shall mean simply that the bookkeeping column vector ${}^{s}X_j$ has a non-zero component in the direction of X_j.†

Since $2k$ is the maximal rank of $[B_\ell]$, k reductions are required in order to reduce $[A_\ell]$ completely (some or all of which may be vacuous). Suppose that $[A_\ell]$ has been reduced s times, where $1 \leq s \leq k$. Then for $c+2 \leq j \leq n$, the bookkeeping column entry on row j of ${}^{s}[A_\ell]$ has the form

$$
{}^{s}X_j = \begin{cases} {}^{s-1}X_j & \text{if } j = L_{i_s}; \\ \left({}^{s-1}b_{i_s,L_{i_s}}\right)\left({}^{s-1}X_j\right) + \left({}^{s-1}b_{j,i_s}\right)\left({}^{s-1}X_{L_{i_s}}\right) & \text{otherwise.} \end{cases}
$$

Here, the index i_s denotes the s-th internal orbit direction, and

$$
\left({}^{s-1}b_{i_s,L_{i_s}}\right)\left({}^{s-1}X_j\right) = \left({}^{s-1}b_{i_s,L_{i_s}}\right) \times \cdots \times \left({}^{1}b_{i_2,L_{i_2}}\right)\left(b_{i_1,L_{i_1}}\right) X_j,
$$

where $b_{i_1,L_{i_1}}, \ldots, {}^{s-1}b_{i_s,L_{i_s}}$ are the polynomials (in the parametrizing components of ℓ) which have been prohibited from being zero by the first s (not necessarily distinct) strong genericity conditions.

The general stage of the construction process may now be described. For each $c+2 \leq j \leq n$, there are exactly three cases to consider. We shall examine each case under the supposition that $[A_\ell]$ has been reduced s times, where $1 \leq s \leq k$.

Case 1 Suppose row j consists entirely of zeros. Then $j \in D$, the set of parameter indices, and $\mathfrak{r}_{\ell j-1} \subset \mathfrak{r}_{\ell j}$ (by Theorem 2.3.1). Hence if

$$
\mathfrak{m}_{\ell j-1} = \mathbb{R}\text{-span}\{X_1, \ldots, X_c, X_{m_1}, X^{\ell}_{m_2}, \ldots, X^{\ell}_{m_{t-1}}\},
$$

† Note that none of the vectors ${}^{s-1}X_j$ depends on on the first $c+1$ vectors $X_1, \ldots, X_{c+1}$ of the basis $\mathcal{B}$ because these vectors were replaced by zeros in the bookkeeping column before reduction was begun.

where $m_1 = c{+}1$ and $X_{m_1}, X^{\ell}_{m_2}, \ldots, X^{\ell}_{m_{t-1}}$ are the former bookkeeping column entries on rows $m_1, \ldots, m_{t-1}$ of ${}^s[A_\ell]$, then we may set

$$\mathfrak{m}_{\ell j} = \mathfrak{m}_{\ell j-1} + \mathbb{R}\, X^{\ell}_{m_t},$$

where $X^{\ell}_{m_t} = {}^sX_j$ is the current bookkeeping column entry on row j of ${}^s[A_\ell]$. We then replace $X^{\ell}_{m_t}$ by zero in the bookkeeping column. However, because ℓ_j is a parametrizing component, we do *not* set it equal to zero. Since row j consists of zeros, no reduction of ${}^s[A_\ell]$ is possible, so if $j < n$, we proceed to row $j+1$; otherwise, we are done.

<u>Case 2</u> Suppose row j contains an entry ${}^sb_{j,L_j}$ which is the leftmost non-trivial polynomial in the parametrizing components of ℓ^j, and which also lies *above* the diagonal of ${}^s[B_\ell]$, that is, $L_j > j$. (Note that this implies that $j < n = \dim \mathfrak{n}$.) Then only zeros appear on that part of row j which is below the diagonal, and so $\mathfrak{r}_{\ell j-1} \subset \mathfrak{r}_{\ell j}$ (by Theorem 2.3.1). If, as in Case 1,

$$\mathfrak{m}_{\ell j-1} = \mathbb{R}\text{-span}\{X_1, \ldots, X_c, X_{m_1}, X^{\ell}_{m_2}, \ldots, X^{\ell}_{m_{t-1}}\},$$

where $m_1 = c{+}1$ and $X_{m_1}, X^{\ell}_{m_2}, \ldots, X^{\ell}_{m_{t-1}}$ are the former bookkeeping column entries on rows $m_1, \ldots, m_{t-1}$ of ${}^s[A_\ell]$, then we may again set

$$\mathfrak{m}_{\ell j} = \mathfrak{m}_{\ell j-1} + \mathbb{R}\, X^{\ell}_{m_t},$$

where $X^{\ell}_{m_t} = {}^sX_j$ is the current bookkeeping column entry on row j of ${}^s[A_\ell]$. Since $X^{\ell}_{m_t} \in \mathfrak{m}_{\ell j}$, m_t is an internal orbit index, so we set $i_{s+1} = m_t = j$. Then we replace ${}^sX_j$ by zero in the bookkeeping column and establish our $(s+1)$-th parametrization condition

$$(PC \quad s+1) \qquad\qquad \ell_{i_{s+1}} = \ell_{L_{i_{s+1}}} = 0,$$

and our $(s+1)$-th strong genericity condition

$$(SGC \quad s+1) \qquad\qquad {}^sb_{j,L_j} \;=\; {}^sb_{i_{s+1},L_{i_{s+1}}} \neq 0.$$

(Even if this entry has appeared in a prior SGC, there is no harm in its appearing in another). Next, we produce the $(s+1)$-times reduced matrix ${}^{s+1}[A_\ell]$ by row- and column-reducing ${}^s[A_\ell]$ using ${}^sb_{i_{s+1},L_{i_{s+1}}}$ and its antisymmetric partner. Finally, we proceed to row $j+1$.

<u>**Case 3**</u> Suppose row j contains an entry ${}^sb_{j,L_j}$ which is the leftmost non-trivial polynomial in the parametrizing components of ℓ^j, and which is *below* the diagonal of ${}^s[B_\ell]$, that is, $L_j < j$. Then this entry has already appeared in a strong genericity condition, and so it is prohibited from being zero for the functionals ℓ that we are considering. As a consequence, the bottom row of the s-times reduced upper left-hand block matrix ${}^s[B_\ell]_{j\times j}$ does not consist of zeros, and so $\mathfrak{r}_{\ell j} \subset \mathfrak{r}_{\ell j-1}$ (by Theorem 2.3.1), and we may set

$$\mathfrak{m}_{\ell j} = \mathfrak{m}_{\ell j-1}.$$

This implies that ${}^sX_j$, the bookkeeping column entry on row j of ${}^s[A_\ell]$, is *not* an element of any of the polarizations $\mathfrak{m}_{\ell j}$, nor will it be an element of the polarizations $\mathfrak{m}_\ell$ when their construction is complete. But—and here is the important part—if ${}^sX_j \notin \mathfrak{m}_\ell$, then neither is X_j, the j-th vector in the original strong Malcev basis $\mathcal{B}$ for $\mathfrak{n}$. We already know the reason: by construction, ${}^{s}X_j$ is a linear combination of the vector X_j and certain other vectors X_i with $i < j$, and the coefficient of X_j in this linear combination is guaranteed by our previous strong genericity conditions never to be zero for the functionals ℓ that we are considering. What all of this means is that the vector X_j is *external* to all of the polarizations that we are constructing.

Now X_j may or may not be the first such external orbit vector from the basis $\mathcal{B}$ found during the construction process. Indeed, suppose that X_j is the w-th one found, where $1 \leq w \leq k$. Then we set $X_{e_w} = X_j$ and we call X_{e_w} the w-th external orbit vector. As noted above, $e_w \in J_E$. Since the entry ${}^{s}b_{e_w,L_{e_w}}$ is below the diagonal, it will have already been subject to a strong genericity condtion and been used to sweep out column L_{e_w} and row e_w. Hence no reduction of ${}^{s}[A_\ell]$ is necessary. Also, ℓ will have already been subject to the parametrization condition $\ell_{e_w} = 0$. So if $j < n$, we proceed to row $j+1$; otherwise, we are done.

§2.10 Results of the General Construction

When the construction process just outlined is completed (i.e., when $j = n$), we find ourselves in the possession of n vectors

$$X_1, \ldots, X_c, X_{m_1}, X^{\ell}_{m_2}, \ldots, X^{\ell}_{m_p}, X_{e_1}, \ldots, X_{e_k}$$

such that the entire Lie algebra

$$\mathfrak{n} = \mathbb{R}\text{-span}\{X_1, \ldots, X_c, X_{m_1}, X^{\ell}_{m_2}, \ldots, X^{\ell}_{m_p}, X_{e_1}, \ldots, X_{e_k}\},$$

and the subspace

$$\mathfrak{m}_\ell = \mathbb{R}\text{-span}\{X_1, \ldots, X_c, X_{m_1}, X^\ell_{m_2}, \ldots, X^\ell_{m_p}\}$$

is a polarizing subalgebra for the linear functional $\ell \in \mathfrak{n}^*$, provided ℓ satisfies the strong genericity condition

$$(SGC\ *) \qquad \left({}^{k-1}b_{i_k,L_{i_k}}\right) \times \cdots \times \left({}^{1}b_{i_2,L_{i_2}}\right)\left(b_{i_1,L_{i_1}}\right) \neq 0$$

and the parametrization conditions

$$(PC\ *) \qquad \ell_{i_1} = \cdots = \ell_{i_k} = \ell_{e_1} = \cdots = \ell_{e_k} = 0.$$

If we call the set of all such functionals $\mathcal{W}_S$, then $(SGC\ *)$ assures us that $\mathcal{W}_S \subset U$, the set of generic functionals with respect to the basis $\mathcal{B}$, and $(PC\ *)$ assures us that $\mathcal{W}_S \subset W_D$. Hence, $\mathcal{W}_S \subset \mathcal{W} = U \cap W_D$, as desired. The set $\mathcal{W}_S$ consists, of course, of our strongly generic parametrizing functionals. A typical member of $\mathcal{W}_S$ has the form

$$\ell = \ell_{d_1} X^*_{d_1} + \cdots + \ell_{d_r} X^*_{d_r},$$

where $1 = d_1 < \cdots < d_r \leq n$. Recall that these indices comprise the set D of dual (or parameter) indices. It is important to note that it is only the components $\ell_{d_1}, \ldots, \ell_{d_r}$ that appear in the condition $(SGC\ *)$.

If we now divide each vector $X^\ell_{m_t}$ by the coefficient of its X_{m_t} term, which coefficient is guaranteed by $(SGC\ *)$ to be non-zero for $\ell \in \mathcal{W}_S$, we find that

$$Y^\ell_{m_t} = X_{m_t} + \sum_{e_j < m_t} c_{m_t,e_j}(\ell)\, X_{e_j},$$

where X_{m_t} is a fixed element of $\mathcal{B}$ and the coefficients $c_{m_t,e_j}(\ell)$ are rational nonsingular functions which depend only on the parametrizing components $\ell_{d_1}, \ldots, \ell_{d_s}$ of ℓ such that $d_s < m_t$. The reason for this last remark is clear: rows $1, \ldots, m_t$ of the matrix $[B_\ell]$ contain entries which depend, at most, on the components $\ell_1, \ldots, \ell_c, \ell_{m_1}, \ldots, \ell_{m_{t-1}}$ of ℓ, and so the bookkeeping column entry on row m_t which gave rise to the vector $Y_{m_t}^\ell$ can itself depend at most on the same components (because of its mode of construction). It is important to observe, too, that the summation in the above expression for $Y_{m_t}^\ell$ involves (besides X_{m_t} itself), only fixed, non-ℓ-dependent external orbit vectors X_{e_j}, with $e_j < m_t$. Again, this is due to our construction process, which involves zeroing out bookkeeping column vectors found to be elements of $\mathfrak{m}_\ell$.

Why is the set $\{X_1, \ldots, X_c, Y_{m_1}, Y_{m_2}^\ell, \ldots, Y_{m_p}^\ell, X_{e_1}, \ldots, X_{e_k}\}$ a weak Malcev basis for $\mathfrak{n}$? In the first place, for each $1 \leq t \leq p$, the subspace

$$\mathbb{R}\text{-span}\{X_1, \ldots, X_c, Y_{m_1}, Y_{m_2}^\ell, \ldots, Y_{m_t}^\ell\}$$

is a subalgebra by construction . And in the second place,

$$\begin{aligned}[Y_{m_t}^\ell, X_{e_i}] &= [X_{m_t} + \sum_{e_j < m_t} c_{m_t,e_j}(\ell)\, X_{e_j}, X_{e_i}] \\ &= [X_{m_t}, X_{e_i}] + \sum_{e_j < m_t} c_{m_t,e_j}(\ell)\, [X_{e_j}, X_{e_i}],\end{aligned}$$

which vector is contained in

$$\mathfrak{m}_\ell \oplus \mathbb{R}\text{-span}\{X_{e_1}, \ldots, X_{e_{i-1}}\}$$

because $\mathcal{B}$ is a strong Malcev basis for $\mathfrak{n}$. Hence for each $1 \leq i \leq k$, we have that

$$\mathfrak{m}_\ell \oplus \mathbb{R}\text{-span}\{X_{e_1}, \ldots, X_{e_i}\}$$

is a subalgebra, and so

$$\mathcal{B}_w(\ell) = \{X_1, \ldots, X_c, Y_{m_1}, Y^{\ell}_{m_2}, \ldots, Y^{\ell}_{m_p}, X_{e_1}, \ldots, X_{e_k}\}$$

is indeed a weak Malcev basis for $\mathfrak{n}$, as desired.

All that remains to be shown is that the Plancherel measure of $\mathcal{W} \sim \mathcal{W}_S$ is zero, so the strongly generic parametrizing functionals induce almost all of the general position irreducible unitary representations in $\widehat{N}$. To this end, let

$$Q(\ell_{d_1}, \ldots, \ell_{d_r}) = \left({}^{k-1}b_{i_k, L_{i_k}}\right) \times \cdots \times \left({}^{1}b_{i_2, L_{i_2}}\right)\left(b_{i_1, L_{i_1}}\right),$$

where the right-hand side is the polynomial in $(SGC\ *)$. If $\ell \in \mathcal{W}_S$, we know that $Q(\ell_{d_1}, \ldots, \ell_{d_r}) \neq 0$. In fact, $\mathcal{W}_S$ is just the Zariski-open set

$$\mathcal{W}_S = \{\ell \in \mathcal{W}: \quad Q(\ell_{d_1}, \ldots, \ell_{d_r}) \neq 0\}.$$

Since the zero set of the non-trivial polynomial $Q(\ell_{d_1}, \ldots, \ell_{d_r})$ is at most $(r-1)$-dimensional, we have that

$$m_r^*(\mathcal{W} \sim \mathcal{W}_S) = 0,$$

where m_r^* is r-dimensional Lebesgue measure on $\mathcal{W}$. But then the Plancherel measure of $\mathcal{W} \sim \mathcal{W}_S$ is

$$\begin{aligned} \rho(\mathcal{W} \sim \mathcal{W}_S) &= \int_{\mathcal{W} \sim \mathcal{W}_S} |\mathrm{Pfaffian}(\ell)|\, dm_r^*(\ell) \\ &= 0, \end{aligned}$$

and we are done.†

---

† If a strong Malcev basis has been fixed for $\mathfrak{n}$, thus determining the set $\{X_{j_1}, \ldots, X_{j_{2k}}\}$ of orbit vectors, the Pfaffian of the functional ℓ is the polynomial function defined by

$$\mathrm{Pffaffian}(\ell)^2 = \det B,$$

where $B_{ik} = B_\ell(X_{j_i}, X_{j_k})$. See [3], pp. 150ff. for further details.

CHAPTER 3

Applying Special Polarizations

§ 3.1 Still More Background

In proving Conjecture 1.1 for the selected groups N mentioned in Chapter 1, we shall need to work with the operator-valued Fourier transform of a suitable function φ. We begin with a general characterization of this transform (see [2], pp. 206–207).

Let $\ell \in \mathfrak{n}^*$, and let $\pi = \pi_\ell$ be the irreducible unitary representation associated with the coadjoint orbit $\mathrm{Ad}^*(N)\,\ell$. If $\mathfrak{m}$ is any fixed polarization for ℓ, we choose a weak Malcev basis $\mathcal{B}_w = \{W_1, \ldots, W_m, U_1, \ldots, U_k\}$ for $\mathfrak{n}$ which passes through $\mathfrak{m}$. Then $M = \exp(\mathfrak{m})$, and $\Sigma = \exp(\mathbb{R}\,U_1) \cdots \exp(\mathbb{R}\,U_k)$ is a closed cross-section of $M \backslash N$. We recall from Chapter 1 that π may be induced from the character $\chi = e^{2\pi i \ell} \circ \log$ on the subgroup M, and that π acts on a Hilbert space $\mathcal{H}_\pi$ of functions f on N that vary like χ along M-cosets, that is, $f(mn) = \chi(m) f(n)$. The action of π is right-translation: $\pi(x) f(n) = f(nx)$ for all $x, n \in N$.

Define polynomial maps

$$\gamma : \mathbb{R}^n \rightarrow N, \quad \alpha : \mathbb{R}^m \rightarrow M, \quad \beta : \mathbb{R}^k \rightarrow N,$$

by

$$\gamma(w,u) = \exp(w_1W_1 + \cdots + w_mW_m)\ \exp(u_1U_1)\cdots\exp(u_kU_k),$$

$$\alpha(w) = \gamma(w,0), \quad \beta(u) = \gamma(0,u),$$

and let dn, dm, and $d\dot{n}$ be the invariant measures on N, M, and $M\backslash N$ determined by the Lebesgue measures $dW\,dU$, dW, and dU. (Of course, α is just the exponential map on $\mathfrak{m}$.) Then, using $d\dot{n}$ in the definition of the norm, $\|f\|^2_{\mathcal{H}_\pi} = \int_{M\backslash N} |f|^2\, d\dot{n}$, the map sending $f \to \tilde{f}(u) = f(\beta(u))$ is an isometry from $\mathcal{H}_\pi$ to $L^2(\mathbb{R}^k, dU)$, and so π may be modelled as a right action on $L^2(\mathbb{R}^k)$.

Let $f \in \mathcal{H}_\pi$ be continuous (so $\tilde{f} \in C(\mathbb{R}^k) \cap L^2(\mathbb{R}^k)$). Then for suitable functions φ (e.g., $\varphi \in C_c^\infty(N)$), the operator

$$\widehat{\varphi}(\pi)(\cdot) = \int_N \varphi(n)\,\pi(n)(\cdot)\,dn$$

produces absolutely convergent integrals when applied to f. Recalling from Chapter 1 that an element $n \in N$ may be written (uniquely) as the product $n = m \cdot \beta(u)$, we have the following standard computation:

$$\begin{aligned}
\big(\widehat{\varphi}(\pi)(f)\big)\big(\beta(t)\big) &= \int_N \varphi(n)\,\pi(n)\,f\big(\beta(t)\big)\,dn \\
&= \int_N \varphi(n)\,f\big(\beta(t)\,n\big)\,dn \\
&= \int_N \varphi\big(\beta(t)^{-1}\,n\big)\,f(n)\,dn \\
&= \int_M \int_{\mathbb{R}^k} \varphi\big(\beta(t)^{-1}\,m\,\beta(u)\big)\,f\big(m\,\beta(u)\big)\,dm\,du \\
&\overset{(5)}{=} \int_{\mathbb{R}^k} \left(\int_M \varphi\big(\beta(t)^{-1}\,m\,\beta(u)\big)\,\chi(m)\,dm \right) f\big(\beta(u)\big)\,du,
\end{aligned}$$

where in equation (5) we have used Fubini's theorem and the χ-covariance of f. The action of the operator-valued Fourier transform on a function f from the representation space $\mathcal{H}_\pi$ is thus given by the k-dimensional integral of the product of f with the kernel function

$$K_\varphi(t,u) = \int_M \varphi\big(\beta(t)^{-1}\, m\, \beta(u)\big)\, \chi(m)\, dm.$$

§3.2 A Proposition

We may now prove a slightly weakened version† of Conjecture 1.1 for a large sub-class of nilpotent Lie groups:

3.2 Proposition. *Let N be a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}$, and let N have the property that the set $\mathcal{W} \subset \mathfrak{n}^*$ of linear functionals which parametrize generic coadjoint orbits (with respect to any fixed strong Malcev basis for $\mathfrak{n}$) is polarized by a single maximal subordinate subalgebra $\mathfrak{m}$. Let $\varphi \in C_c^\infty(N)$ and suppose that $\widehat{\varphi}(\pi) = 0$ for all $\pi \in E \subset \widehat{N}_{g.p.}$, where the Plancherel measure $\rho(E) > 0$. Then $\varphi \equiv 0$.*

Proof. As in Chapter 2, let $\mathcal{B} = \{X_1, \ldots, X_n\}$ be a strong Malcev basis which passes through the center $\mathfrak{z}(\mathfrak{n})$ of $\mathfrak{n}$, and let

$$\mathcal{B}_w = \{X_1, \ldots, X_c, Y_{m_1}, \ldots, Y_{m_p}, X_{e_1}, \ldots, X_{e_k}\}$$

† We assume that φ is in $C_c^\infty(N)$, rather than $L_c^\infty(N)$, in order to use the nilpotent Plancherel theorem. The two examples which follow this proposition do *not* assume that φ is smooth.

be the corresponding special weak Malcev basis (note that $Y_{m_2}, \ldots, Y_{m_p}$ do not depend on any elements of $\mathfrak{n}^*$ in the present case). For convenience in subscripting, let us set $W_1 = X_1, \ldots, W_c = X_c, W_{c+1} = Y_{m_1}, \ldots, W_m = Y_{m_p}$. Now for each $\pi \in E$, there exists a unique $\ell \in \mathcal{W}$ such that $\pi = \pi_\ell$ (Kirillov). Let $E' \subset \mathcal{W}$ correspond to $E \subset \widehat{N}_{g.p.}$. Then for each $\ell \in E'$, we have (by hypothesis) that $\widehat{\varphi}(\pi_\ell) = 0$. Hence for each continuous $f \in \mathcal{H}_{\pi_\ell} \cong \mathcal{H}_{\widehat{N}_{g.p.}}$, the fixed modelling space for all $\pi \in \widehat{N}_{g.p.}$, we have, as we saw in § 3.1,

$$\begin{aligned} 0 &= \big(\widehat{\varphi}(\pi_\ell)f\big)\big(\beta(t)\big) \\ &= \int_{\mathbb{R}^k} \left(\int_M \varphi\big(\beta(t)^{-1}\, m\, \beta(u)\big)\; e^{2\pi i \ell \left(\log\ m\right)}\, dm \right) f\big(\beta(u)\big)\, du \end{aligned}$$

for all $t \in \mathbb{R}^k$. Since $C(\mathbb{R}^k) \cap L^2(\mathbb{R}^k)$ is dense in $L^2(\mathbb{R}^k)$, it follows that

$$\begin{aligned} 0 &= K_\varphi(t, u, \ell) \\ &= \int_M \varphi\big(\beta(t)^{-1}\, m\, \beta(u)\big)\; e^{2\pi i \ell \left(\log\ m\right)}\, dm \end{aligned}$$

for all $t, u \in \mathbb{R}^k$ and for all $\ell \in E'$.

Now fix $t, u \in R^k$ arbitrarily, and recall that the set $\{1, \ldots, m\}$ of polarization indices may be viewed as the disjoint union of the sets $D = \{d_1, \ldots, d_r\}$ and $J_I = \{i_1, \ldots, i_k\}$. If $\ell = \sum_{i=1}^r \ell_{d_i} W_{d_i}^* \in E'$ and $W = \sum_{j=1}^m w_j W_j \in \mathfrak{m}$, then for all $\ell \in E'$, we have

$$\begin{aligned} 0 &= \int_M \varphi\big(\beta(t)^{-1}\, m\, \beta(u)\big)\; e^{2\pi i \ell \left(\log\ m\right)}\, dm \\ &= \int_{\mathbb{R}^m} \varphi\big(\beta(t)^{-1}\, \alpha(W)\, \beta(u)\big)\; e^{2\pi i \ell \left(\log\ \alpha(W)\right)}\, dW \\ &\overset{(3)}{=} \int_{\mathbb{R}^r} \left(\int_{\mathbb{R}^k} \varphi\big(\beta(t)^{-1}\, \alpha(W)\, \beta(u)\big)\; dw_{i_1} \cdots dw_{i_k} \right) \times \\ &\qquad\qquad\qquad e^{2\pi i \left(w_{d_1}\ell_{d_1} + \cdots + w_{d_r}\ell_{d_r}\right)}\, dw_{d_1} \cdots dw_{d_r}\ . \end{aligned}$$

The inner integral in equation (3) is compactly supported and *independent* of ℓ. Hence, by the Paley-Wiener theorem, the integral over $\mathbb{R}^r$ extends to an entire function on $\mathbb{C}^r$. Since E' has positive r-dimensional measure and the integral vanishes for all $\ell \in E'$, it must in fact vanish for all $\ell \in \mathcal{W}$. Moreover, since t and u were chosen arbitrarily from $\mathbb{R}^k$, we have that the kernel $K_\varphi(t,u,\ell)$ vanishes for all $t,u \in \mathbb{R}^k$ and for all $\ell \in \mathcal{W}$. But this is just to say that $\widehat{\varphi}(\pi)$ is the zero operator.

So we see that $\widehat{\varphi}(\pi) = 0$ for all $\pi \in E$ implies that $\widehat{\varphi}(\pi) = 0$ for all $\pi \in \widehat{N}_{g.p.}$. Now by the nilpotent incarnation of the Plancherel Theorem ([3], pp. 144–161), we find that

$$\begin{aligned} \| \varphi \|_2^2 &= \int_{\widehat{N}} \operatorname{Tr}\big(\widehat{\varphi}(\pi)\ \widehat{\varphi}(\pi)^*\big)\, \mathrm{d}\rho(\pi) \\ &= 0. \end{aligned}$$

Hence $\varphi \equiv 0$, as desired. $\square$

We should note that in their paper "Fourier transforms of smooth functions on certain nilpotent Lie groups" [1], Corwin and Greenleaf show that the kernel K_φ^ℓ can be rewritten as an exponential factor multiplied by the partial Fourier transform (in the first m variables) of $\varphi \circ \gamma$ composed with a polynomial change of variables map, ***provided*** $\varphi \in \mathcal{S}(N)$ and N has the property that every irreducible unitary representation in the support of the Plancherel measure is induced from a single polarizing *ideal*. They point out that many groups possess this latter property; for example, the $(2n+1)$-dimensional Heisenberg groups, and the groups of upper triangular $n \times n$ matrices with 1's on the diagonal. Our proposition is slightly more general, in that we do not require the single polarization to be an ideal, but it is also slightly less general, in that we do not obtain a formula for the kernel K_φ, but rather conclude from its vanishing that the compactly supported, infinitely differentiable function φ must also vanish on N. In case the reader wonders whether there *are* any nilpotent Lie groups with the property assumed in Proposition 3.2, he is referred to the groups labelled $G_{6,12}$ and $G_{6,14}$ in Nielsen ([7], pp. 63ff. and pp. 73ff., respectively).

§ 3.3 Two Examples

We said in Chapter 1 that there are some particular examples of groups lying outside the class just discussed for which Conjecture 1.1 holds as stated. We turn now to two such examples (the writer has dealt similarly with others).

3.3.1 Example. Let $\mathfrak{f}_{2,3}$ denote the 3-step free nilpotent Lie algebra on the 2 generators X_5 and X_4, and let $F_{2,3}$ denote the corresponding 1-connected free nilpotent Lie group. A strong Malcev basis for $\mathfrak{f}_{2,3}$ is given by $\mathcal{B} = \{X_1, \dots, X_5\}$, with non-zero brackets generated by

$$[X_5, X_4] = X_3, \quad [X_5, X_3] = X_2, \quad [X_4, X_3] = X_1.$$

Suppressing central zeros (as in Chapter 2), we may write the augmented matrix $[A_\ell]$ of the bilinear form B_ℓ for $\ell \in \mathfrak{f}_{2,3}{}^*$ as

$$[A_\ell] = \left[\begin{array}{ccc|c} 0 & -\ell_1 & -\ell_2 & X_3 \\ \ell_1 & 0 & -\ell_3 & X_4 \\ \ell_2 & \ell_3 & 0 & X_5 \end{array}\right].$$

Since row 3 is the first non-central row, we note that $X_3 \in \mathfrak{m}_{\ell^3}$. Next, we replace X_3 by zero in the bookkeeping column, set $\ell_1 \neq 0$, and set $\ell_3 = \ell_4 = 0$. Then, using ℓ_1 and $-\ell_1$, we row-reduce and antisymmetrize, producing the matrix

$$ {}^1[A_\ell] = \begin{bmatrix} 0 & -\ell_1 & 0 & | & 0 \\ \ell_1 & 0 & 0 & | & X_4 \\ 0 & 0 & 0 & | & {}^1X_5 \end{bmatrix}, $$

where ${}^1X_5 = \ell_1 X_5 - \ell_2 X_4$. The strongly generic parametrizing functionals in $\mathfrak{f}_{2,3}{}^*$ comprise the set $\mathcal{W}_S = \{\ell \in \mathfrak{f}_{2,3}{}^* : \ell_1 \neq 0,\ \ell_3 = \ell_4 = 0\}$. A polarization for $\ell \in \mathcal{W}_S$ is given by $\mathfrak{m}_\ell = \mathbb{R}\text{-span}\{X_1, X_2, Y_3, Y_5^\ell\}$, where $Y_3 = X_3$ and

$$ \begin{aligned} Y_5^\ell &= \frac{1}{\ell_1}\, {}^1X_5 \\ &= X_5 - \frac{\ell_2}{\ell_1} X_4 \\ &= X_5 + c(\ell)\, X_4. \end{aligned} $$

$\mathfrak{f}_{2,3}$ itself may be written $\mathfrak{m}_\ell \oplus \mathbb{R}\, X_4$.

We have just seen that many nilpotent Lie algebras have the property that a single ideal polarizes all generic functionals. Although we did not mention it above, such an ideal must also be abelian ([1], p. 205). $\mathfrak{f}_{2,3}$ is the lowest dimensional example (and, up to isomorphism, the only 5-dimensional example) to *lack* this property (see [7], Ch. 1). In fact, $\mathfrak{m}_\ell$ is an ideal (as can always be arranged for 3-step algebras), but it is non-abelian and rotates as ℓ varies through the set $\mathcal{W}_S$ of strongly generic parametrizing functionals.

To fix notation, let $W(\ell) = w_1 X_1 + w_2 X_2 + w_3 Y_3 + w_5 Y_5^\ell$ denote an arbitrary element of the polarization $\mathfrak{m}_\ell$, let $m(\ell) = \exp\big(W(\ell)\big)$ denote the corresponding group element, and let $\beta(t) = \exp(tX_4)$. As Euclidean measure on the polarizations $\mathfrak{m}_\ell$ we shall always choose

$$dW = dw_1\, dw_2\, dw_3\, dw_5,$$

which is just the fixed Euclidean measure on the projection of $\mathfrak{m}_\ell$ onto the hyperplane $\mathbb{R}$-span$\{X_1, X_2, X_3, X_5\}$. As remarked in Chapter 2, this choice of measure is valid for all functionals $\ell \in \mathcal{W}_S$ (indeed, for all generic functionals in the present example). The measure $dW\, dx_4$ on $\mathfrak{f}_{2,3}$, with dx_4 being Lebesgue measure on $\mathbb{R}X_4$, corresponds to the invariant (Haar) measure $dm(\ell)\, du$ on the group $F_{2,3}$.

We are now ready to state and prove Conjecture 1.1 for $F_{2,3}$:

Let $\varphi \in L_c^\infty(F_{2,3})$, and suppose that $\widehat{\varphi}(\pi) = 0$ for all $\pi \in E$, where E is a subset of the strongly generic representations in $\widehat{F}_{2,3}$ and E has positive Plancherel measure. Then $\varphi = 0$ almost everywhere on $F_{2,3}$.

Proof. For each $\pi \in E$, there exists a unique $\ell \in \mathcal{W}_S$ such that $\pi = \pi_\ell$. Letting $E' \subset \mathcal{W}_S$ correspond to $E \subset \widehat{F}_{2,3}$, we have by hypothesis that $\widehat{\varphi}(\pi_\ell) = 0$ for all $\ell \in E'$. Hence, for each continuous $f \in \mathcal{H}_{\pi_\ell} \cong L^2(\mathbb{R})$, we have (writing f for the function we earlier called $\tilde{f}$):

$$\begin{aligned} 0 &= \big(\widehat{\varphi}(\pi_\ell) f\big)(t) \\ &= \int_{\mathbb{R}} \left(\int_{M_\ell} \varphi\big(\beta(t)^{-1}\, m(\ell)\, \beta(u)\big)\, e^{2\pi i \ell \left(\log\, m(\ell)\right)}\, dm(\ell) \right) f(u)\, du \end{aligned}$$

for all $t \in \mathbb{R}$, and for all $\ell \in E'$. Since $C(\mathbb{R}) \cap L^2(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, we must therefore have

$$0 = \int_{M_\ell} \varphi\big(\beta(t)^{-1}\, m(\ell)\, \beta(u)\big)\, e^{2\pi i \ell \left(\log\, m(\ell)\right)}\, dm(\ell)$$

for all $t, u \in \mathbb{R}$, and for all $\ell \in E'$.

When we reached this point in the proof of Proposition 3.2, we were able to continue by noting that for each fixed choice of $t, u \in \mathbb{R}^k$, and for all $\ell \in E'$, we have $0 = \int_M \varphi(\beta(t)^{-1} m \beta(u))\, e^{2\pi i \ell (\log\ m)}\, dm$, and so on. But there is an important difference between then and now: the present integral is over M_ℓ, not over M. As ℓ varies through E', M_ℓ will also vary. We must find a new argument for the vanishing of φ.

Since we shall need to manipulate the kernel function K_φ, we employ the notational conveniences

$$X * Y = \log\big(\exp(X) \cdot \exp(Y)\big) \qquad \text{(Campbell-Baker-Hausdorff product)}$$

and $\tilde{\varphi} = \varphi \circ \exp$. Then for any $\ell \in \mathcal{W}_S$, we have:

$$\begin{aligned}
K_\varphi^\ell(t,u) &= K_\varphi(t,u,\ell) \\
&= \int_{M_\ell} \varphi\big(\beta(t)^{-1}\ m(\ell)\ \beta(u)\big)\ e^{2\pi i \ell \left(\log\ m(\ell)\right)}\ dm(\ell) \\
&= \int_{\mathfrak{m}_\ell} \tilde{\varphi}\big(-tX_4 * W(\ell) * uX_4\big)\ e^{2\pi i \ell \left(\log(\exp\ W(\ell))\right)}\ dW \\
&= \int_{\mathfrak{m}_\ell} \tilde{\varphi}\big(-tX_4 * W(\ell) * tX_4 * (u-t)X_4\big)\ e^{2\pi i \ell \left(W(\ell)\right)}\ dW \\
&\overset{(5)}{=} \int_{\mathfrak{m}_\ell} \tilde{\varphi}\big(W(\ell) * (u-t)X_4\big)\ e^{2\pi i \ell \left(tX_4 * W(\ell) * -tX_4\right)}\ dW \\
&= \int_{\mathfrak{m}_\ell} \tilde{\varphi}\big(W(\ell) * (u-t)X_4\big)\ e^{2\pi i\, \mathrm{Ad}^*(\exp\ -tX_4)\ell \left(W(\ell)\right)}\ dW,
\end{aligned}$$

where in equation (5) we use the fact that $\mathfrak{m}_\ell$ is an ideal in $\mathfrak{f}_{2,3}$.

Writing $\tilde{\ell} = \mathrm{Ad}^*(\exp -tX_4)\ell$, we compute that

$$\begin{aligned}\tilde{\ell} &= \ell + \mathrm{ad}^*(-tX_4)\ell + \tfrac{1}{2}\big(\mathrm{ad}^*(-tX_4)\big)^2(\ell) \\ &= \ell_1X_1^* + \ell_2X_2^* + (t\ell_1)X_3^* + (\ell_5 - \tfrac{1}{2}t^2\ell_1)X_5^*,\end{aligned}$$

whence

$$\tilde{\ell}\big(W(\ell)\big) = \ell_1w_1 + \ell_2w_2 + (t\ell_1)w_3 + (\ell_5 - \tfrac{1}{2}t^2\ell_1)w_5.$$

Also, using the C-B-H formula, we find that

$$\begin{aligned}&W(\ell) * (u-t)X_4 \\ &= \big(w_1X_1 + w_2X_2 + w_3X_3 + w_5Y_5^{\ell}\big) * (u-t)X_4 \\ &= \big(w_1X_1 + w_2X_2 + w_3X_3 + w_5X_5 + w_5c(\ell)X_4\big) * (u-t)X_4 \\ &= \big(w_1X_1 + \cdots + w_5X_5 + w_5c(\ell)X_4\big) * \big(-w_5c(\ell)\big)X_4 * \big(w_5c(\ell)\big)X_4 * (u-t)X_4 \\ &\overset{(4)}{=} \big((w_1+P_1)X_1 + (w_2+P_2)X_2 + (w_3+P_3)X_3 + w_5X_5\big) * \big(u-t+w_5c(\ell)\big)X_4 \\ &\overset{(5)}{=} \big(w_1+P_1, w_2+P_2, w_3+P_3, w_5\,;\, u-t+w_5c(\ell)\big),\end{aligned}$$

where the notational abbreviation in equation (5) is introduced for typographical convenience. The polynomials P_1, P_2 and P_3 in equation (4) are easily computed:

$$\begin{aligned}P_3 &= -\tfrac{1}{2}w_5^2c(\ell), \\ P_2 &= -\tfrac{1}{12}w_5^3c(\ell), \\ P_1 &= \tfrac{1}{2}w_3w_5c(\ell)) - \tfrac{1}{6}w_5^3c(\ell)^2.\end{aligned}$$

Returning to the computation of K^{ℓ}_{φ}, we see that

$$
\begin{aligned}
&K_{\varphi}(t,u,\ell) \\
&= \int_{\mathfrak{m}_\ell} \tilde{\varphi}\big(W(\ell) * (u-t)X_4\big)\ e^{2\pi i\,\tilde{\ell}\left(W(\ell)\right)}\ dW \\
&= \int_{\mathbb{R}^4} \tilde{\varphi}\big(w_1+P_1, w_2+P_2, w_3+P_3, w_5\,;\ u-t+w_5c(\ell)\big) \times \\
&\qquad\qquad e^{2\pi i\left(\ell_1w_1+\ell_2w_2+(t\ell_1)w_3+(\ell_5-\frac{1}{2}t^2\ell_1)w_5\right)}\ dw_1\ dw_2\ dw_3\ dw_5 \\
&\overset{(3)}{=} \int_{\mathbb{R}} \Bigg(\int_{\mathbb{R}^3} \tilde{\varphi}\big(w_1+P_1, w_2+P_2, w_3+P_3, w_5\,;\ u-t+w_5c(\ell)\big) \times \\
&\qquad\qquad e^{2\pi i\left(\ell_1w_1+\ell_2w_2+(t\ell_1)w_3+(-\frac{1}{2}t^2\ell_1)w_5\right)}\ dw_1\ dw_2\ dw_3\Bigg)\ e^{2\pi i\,(\ell_5w_5)}\ dw_5 \\
&= \int_{\mathbb{R}} {}^1K_{\tilde{\varphi}}\big(\ell_1,\ell_2,t,w_5\,;\ u-t+w_5c(\ell)\big)\ e^{2\pi i\,(\ell_5w_5)}\ dw_5,
\end{aligned}
$$

where ${}^1K_{\tilde{\varphi}}\big(\ell_1,\ell_2,t,w_5\,;\ u-t+w_5c(\ell)\big)$ is a handy (and suggestive) way of referring to the inner 3-dimensional integral on the right-hand side of equation (3).

We observe that this abridged kernel ${}^1K_{\tilde{\varphi}}\big(\ell_1,\ell_2,t,w_5\,;\ u-t+w_5c(\ell)\big)$ is *independent* of ℓ_5 and compactly supported in the variable w_5. Indeed, because of these facts, the hypotheses of the Paley-Wiener theorem for $\mathbb{R}$ are satisfied, and so the integral

$$
(*)\qquad \int_{\mathbb{R}} {}^1K_{\tilde{\varphi}}(\ell_1,\ell_2,t,w_5\,;\ u-t+w_5c(\ell))\ e^{2\pi i\,(\ell_5w_5)}\ dw_5
$$

defines a function of ℓ_5 (a partial Fourier transform of the abridged kernel) which extends to an entire function on $\mathbb{C}$. We shall use this fact in a moment.

By hypothesis, for each $\ell \in E'$, the integral $(*)$ is zero for all $t, u \in \mathbb{R}$. If we put $a = u - t + w_5 c(\ell)$, we note that a can be fixed arbitrarily by simply letting $u = a + t - w_5 c(\ell)$. In effect, the degree of freedom represented by the variable u absorbs all variation in t, w_5 and $c(\ell)$, thus *fixing* a. Hence, for each $\ell \in E'$, the integral $(*)$ is zero for all $t, a \in \mathbb{R}$. Again, if we set $\tilde{\ell}_3 = t\ell_1$, we see that $(*)$ vanishes for all $\tilde{\ell}_3 \in \mathbb{R}$, using our freedom in t to vary $\tilde{\ell}_3$.† So we see that for each $\ell \in E'$, the integral $(*)$ is zero for all $\tilde{\ell}_3, a \in \mathbb{R}$.

Let us now consider ℓ to be the ordered triple (ℓ_1, ℓ_2, ℓ_5). Because E' has positive 3-dimensional measure, there exists a set $E'_{1,2}$ (say) of positive 2-dimensional measure such that for each pair $(\ell_1, \ell_2) \in E'_{1,2}$, the triple (ℓ_1, ℓ_2, ℓ_5) is an element of E' for all ℓ_5 in a set $E'_5(\ell_1, \ell_2)$ of positive 1-dimensional measure. (This is a consequence of Fubini's theorem.)

Fix the pair (ℓ_1, ℓ_2) arbitrarily in the set $E'_{1,2}$, and then fix $\tilde{\ell}_3$ and a arbitrarily in $\mathbb{R}$. For all ℓ_5 in the set $E'_5(\ell_1, \ell_2)$, we have that

$$0 = \int_{\mathbb{R}} {}^1K_{\tilde{\varphi}}(\ell_1, \ell_2, \tilde{\ell}_3, w_5\,;\, a)\; e^{2\pi i\,(\ell_5 w_5)}\; dw_5.$$

Because this integral has an entire extension and vanishes on a set of positive 1-dimensional measure, it vanishes for *all* $\ell_5 \in \mathbb{R}$. But this vanishing for all ℓ_5 implies, in turn, that

† In fact, $\tilde{\ell}_3$ is a kind of ersatz ℓ_3, making up for the absence of the X_3^*-component in the strongly generic parametrizing functionals ℓ with which we are working. We might also point out that, strictly speaking, one ought now to replace every occurrence of t in the kernel by $\frac{\tilde{\ell}_3}{\ell_1}$.

$$0 = {}^1K_{\tilde{\varphi}}(\ell_1, \ell_2, \tilde{\ell}_3, w_5\,;\,a)$$

$$= e^{2\pi i(-\frac{1}{2}t^2\ell_1 w_5)} \int\limits_{\mathbb{R}^3} \tilde{\varphi}\big(w_1 + P_1, w_2 + P_2, w_3 + P_3, w_5\,;\,a\big) \times$$

$$e^{2\pi i\left(\ell_1 w_1 + \ell_2 w_2 + \tilde{\ell}_3 w_3\right)}\; dw_1 dw_2 dw_3,$$

or, cancelling the non-zero factor $e^{2\pi i(-\frac{1}{2}t^2\ell_1 w_5)}$,

$$0 = \int\limits_{\mathbb{R}^3} \tilde{\varphi}\big(w_1 + P_1, w_2 + P_2, w_3 + P_3, w_5\,;\,a\big)\; e^{2\pi i\left(\ell_1 w_1 + \ell_2 w_2 + \tilde{\ell}_3 w_3\right)}\; dw_1 dw_2 dw_3$$

for each fixed pair $(\ell_1, \ell_2) \in E'_{1,2}$, and for each fixed choice of $\tilde{\ell}_3, a, w_5 \in \mathbb{R}$.

Since the polynomial P_3 does not depend on the remaining variables of integration, the translation invariance of Lebesgue measure implies that sending $w_3 \longmapsto w_3 - P_3$ in this last integral does not affect its vanishing. Making this replacement and cancelling the resulting non-zero factor $e^{2\pi i(-\tilde{\ell}_3 P_3)}$, we find that the polynomial P_1 becomes

$${}^1P_1 = \tfrac{1}{2}w_3 w_5 c(\ell) + \tfrac{1}{12}w_5^3 c(\ell)^2,$$

and the last integral above becomes

$$\int\limits_{\mathbb{R}^3} \tilde{\varphi}\big(w_1 + {}^1P_1, w_2 + P_2, w_3, w_5\,;\,a\big)\; e^{2\pi i\left(\ell_1 w_1 + \ell_2 w_2 + \tilde{\ell}_3 w_3\right)}\; dw_1 dw_2 dw_3$$

$$\overset{(1)}{=} \int\limits_{\mathbb{R}} \Bigg(\int\limits_{\mathbb{R}^2} \tilde{\varphi}\big(w_1 + {}^1P_1, \ldots\,;\,a\big) e^{2\pi i\left(\ell_1 w_1 + \ell_2 w_2\right)}\; dw_1 dw_2 \Bigg) e^{2\pi i\,\tilde{\ell}_3 w_3}\; dw_3$$

$$= \int\limits_{\mathbb{R}} {}^2K_{\tilde{\varphi}}(\ell_1, \ell_2, w_3, w_5\,;\,a)\; e^{2\pi i\,(\tilde{\ell}_3 w_3)}\; dw_3,$$

where $^2K_{\tilde{\varphi}}(\ell_1, \ell_2, w_3, w_5\,;\,a)$ refers to the inner 2-dimensional integral in equation (1) above.

This twice-abridged kernel $^2K_{\tilde{\varphi}}(\ell_1, \ell_2, w_3, w_5\,;\,a)$ is *independent* of $\tilde{\ell}_3$ (and so of t), and is compactly supported in the variable w_3. Because of these facts, the integral—call it $(**)$— on the right-hand side of equation (1) defines a function of $\tilde{\ell}_3$ (a partial Fourier transform of the twice-abridged kernel) which extends to an entire function on $\mathbb{C}$.

Now we argue almost as before. Fix the pair $(\ell_1, \ell_2) \in E'_{1,2}$, and also fix $a, w_5 \in \mathbb{R}$. The integral $(**)$ vanishes for all $\tilde{\ell}_3 \in \mathbb{R}$, which then implies (again, by Paley-Wiener) that

$$
\begin{aligned}
0 &= {}^2K_{\tilde{\varphi}}(\ell_1, \ell_2, w_3, w_5\,;\,a) \\
&= \int_{\mathbb{R}^2} \tilde{\varphi}(w_1 + {}^1P_1, w_2 + P_2, w_3, w_5\,;\,a) e^{2\pi i \left(\ell_1 w_1 + \ell_2 w_2\right)}\, dw_1 dw_2
\end{aligned}
$$

for each fixed pair $(\ell_1, \ell_2) \in E'_{1,2}$, and for each fixed $a, w_3, w_5 \in \mathbb{R}$.

Since the polynomials $^1P_1$ and P_2 do not depend on the variables w_1 and w_2, the translation invariance of Lebesgue measure again assures us that sending $w_1 \longmapsto w_1 - {}^1P_1$ and $w_2 \longmapsto w_2 - P_2$ in this last integral does not affect its vanishing. Making these replacements and cancelling the resulting non-zero factor $e^{2\pi i(-\ell_1\, ^1P_1 - \ell_2 P_2)}$, we find that

$$
0 = \int_{\mathbb{R}^2} \tilde{\varphi}(w_1, w_2, w_3, w_5\,;\,a) e^{2\pi i \left(\ell_1 w_1 + \ell_2 w_2\right)}\, dw_1 dw_2
$$

for each choice of $w_3, w_5, a \in \mathbb{R}$, provided the pair (ℓ_1, ℓ_2) is chosen from the set $E'_{1,2}$ of positive 2-dimensional measure. But this last integral is just the partial Fourier transform of $\tilde{\varphi}$ in the central variables w_1 and w_2, and so the fact that it vanishes on $E'_{1,2}$ implies that it vanishes for all pairs (ℓ_1, ℓ_2). Hence the function $\tilde{\varphi} = \varphi \circ \exp$ is zero for almost all $w_1, w_2 \in \mathbb{R}$ for each choice of $w_3, w_5, a \in \mathbb{R}$. We see finally, then, that $\varphi = 0$ almost everywhere on $F_{2,3}$, as desired. $\square$

3.3.2 Example. Let $\mathfrak{n}_7$ denote the 7-dimensional 5-step nilpotent Lie algebra with non-zero brackets generated by

$$[X_7, X_6] = X_5, \quad [X_7, X_5] = X_4, \quad [X_7, X_4] = X_3, \quad [X_7, X_3] = X_2$$

$$[X_6, X_5] = X_3, \quad [X_6, X_4] = X_2, \quad [X_6, X_3] = X_1, \quad [X_5, X_4] = -X_1,$$

and let N_7 denote the corresponding 1-connected nilpotent Lie group. Suppressing central zeros, we may write the augmented matrix $[A_\ell]$ of the bilinear form B_ℓ for $\ell \in \mathfrak{n}_7^*$ as

$$[A_\ell] = \left[\begin{array}{ccccc|c} 0 & 0 & 0 & -\ell_1 & -\ell_2 & X_3 \\ 0 & 0 & \ell_1 & -\ell_2 & -\ell_3 & X_4 \\ 0 & -\ell_1 & 0 & -\ell_3 & -\ell_4 & X_5 \\ \ell_1 & \ell_2 & \ell_3 & 0 & -\ell_5 & X_6 \\ \ell_2 & \ell_3 & \ell_4 & \ell_5 & 0 & X_7 \end{array}\right].$$

Working as before, we find that the strongly generic parametrizing functionals in $\mathfrak{n}_7^*$ comprise the set $\mathcal{W}_S = \{\ell \in \mathfrak{n}_7^* : \ \ell_1 \neq 0, \ \ell_3 = \ell_4 = \ell_5 = \ell_6 = 0\}$. A polarization for $\ell \in \mathcal{W}_S$ is given by $\mathfrak{m}_\ell = \mathbb{R}\text{-span}\{X_1, X_2, Y_3, Y_4, Y_7^\ell\}$, where $Y_7^\ell = X_7 - \frac{\ell_2}{\ell_1} X_6 = X_7 + c(\ell) X_6$. The entire algebra $\mathfrak{n}_7$ may be written as $\mathfrak{m}_\ell \oplus \mathbb{R}\text{-span}\{X_5, X_6\}$.

The polarization $\mathfrak{m}_\ell$ for $\ell \in \mathcal{W}_S$ is *neither* abelian *nor* an ideal. In addition, the two external orbit vectors X_5 and X_6 do not commute with respect to the Lie bracket; indeed, their bracket $[X_6, X_5] = X_3$ is *non-central.* In some respects, then, the algebra $\mathfrak{n}_7$ is more typical than $\mathfrak{f}_{2,3}$, and so it is interesting to see that our conjecture holds in $\mathfrak{n}_7$ as well (although the argument is complicated by the fact that $\mathfrak{m}_\ell$ is not an ideal!).

To fix notation, let $W(\ell) = w_1X_1 + w_2X_2 + w_3Y_3 + w_4Y_4 + w_7Y_7^\ell$ denote an arbitrary element of the polarization $\mathfrak{m}_\ell$, let $m(\ell) = \exp(W(\ell))$ denote the corresponding group element, and let $\beta(u_1, u_2) = \exp(u_1X_5)\exp(u_2X_6)$. As Euclidean measure on the polarizations $\mathfrak{m}_\ell$ we shall always choose

$$dW = dw_1\, dw_2\, dw_3\, dw_4\, dw_7,$$

which is just the fixed Euclidean measure on the projection of $\mathfrak{m}_\ell$ onto the hyperplane $\mathbb{R}$-span$\{X_1, X_2, X_3, X_4, X_7\}$. As before, this choice of measure is valid for all functionals $\ell \in \mathcal{W}_S$. The measure $dW\, dx_5\, dx_6$ on $\mathfrak{n}_7$, with $dx_5\, dx_6$ being Lebesgue measure on $\mathbb{R}$-span$\{X_5, X_6\}$, corresponds to the invariant (Haar) measure $dm(\ell)\, du_1\, du_2$ on the group N_7.

So much being said, we now state and prove Conjecture 1.1 for N_7:

Let $\varphi \in L_c^\infty(N_7)$, and suppose that $\widehat{\varphi}(\pi) = 0$ for all $\pi \in E$, where E is a subset of the strongly generic representations in $\widehat{N}_7$ and E has positive Plancherel measure. Then $\varphi = 0$ almost everywhere on N_7.

Proof. For each $\pi \in E$, there exists a unique $\ell \in \mathcal{W}_S$ such that $\pi = \pi_\ell$. Letting $E' \subset \mathcal{W}_S$ correspond to $E \subset \widehat{N}_7$, we have by hypothesis that $\widehat{\varphi}(\pi_\ell) = 0$ for all $\ell \in E'$. Hence, for each continuous $f \in \mathcal{H}_{\pi_\ell} \cong L^2(\mathbb{R}^2)$, we have (again writing f instead of $\tilde{f}$):

$$\begin{aligned} 0 &= (\widehat{\varphi}(\pi_\ell)f)(t_1, t_2) \\ &= \int_{\mathbb{R}^2} \left(\int_{M_\ell} \varphi(\beta(t_1, t_2)^{-1} m(\ell)\, \beta(u_1, u_2))\, e^{2\pi i \ell\left(\log\ m(\ell)\right)}\, dm(\ell) \right) f(u_1, u_2)\, du_1 du_2 \end{aligned}$$

for all $(t_1, t_2) \in \mathbb{R}^2$, and for all $\ell \in E'$. Since $C(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ is dense in $L^2(\mathbb{R}^2)$, it follows that

$$0 = \int\limits_{M_\ell} \varphi(\beta(t_1, t_2)^{-1}\ m(\ell)\ \beta(u_1, u_2))\ e^{2\pi i \ell \left(\log\ m(\ell)\right)}\ dm(\ell)$$

for all $t_1, t_2, u_1, u_2 \in \mathbb{R}$, and for all $\ell \in E'$.

Reverting to the notational conveniences $X * Y = \log\bigl(\exp(X) \cdot \exp(Y)\bigr)$ and $\tilde{\varphi} = \varphi \circ \exp$, we have for any $\ell \in \mathcal{W}_S$:

$$\begin{aligned}
&K_\varphi^\ell(t_1, t_2, u_1, u_2) \\
&= K_\varphi(t_1, t_2, u_1, u_2, \ell) \\
&= \int\limits_{M_\ell} \varphi(\beta(t_1, t_2)^{-1}\ m(\ell)\ \beta(u_1, u_2))\ e^{2\pi i \ell \left(\log\ m(\ell)\right)}\ dm(\ell) \\
&= \int\limits_{\mathfrak{m}_\ell} \tilde{\varphi}\bigl(-t_2 X_6 * -t_1 X_5 * W(\ell) * u_1 X_5 * u_2 X_6\bigr)\ e^{2\pi i \ell \left(W(\ell)\right)}\ dW.
\end{aligned}$$

Our strategy now is to use the Campbell-Baker-Hausdorff formula 'softly' (i.e., with little regard for the rational coefficients or their signs) to coax the vector

$$-t_2 X_6 * -t_1 X_5 * W(\ell) * u_1 X_5 * u_2 X_6$$

into a form more suited to the purposes of our argument. The form we seek is

$$\widetilde{W}(\ell, t_1, t_2, u_1 - t_1) * (u_1 - t_1 + P_5) X_5 * (u_2 - t_2 + P_6) X_6,$$

where

$$\widetilde{W}(\ell, t_1, t_2, u_1 - t_1) = (w_1 + P_1) X_1 + \cdots + (w_4 + P_4) X_4 + w_7 X_7$$

and the polynomials $P_1, \ldots, P_6$ have certain special dependencies and *non*-dependencies which permit our arguments to proceed. For typographical convenience, we shall write W instead of $W(\ell)$ for a while.

We first do the computation in 'scroll' mode, without stopping for comments. The underline delineates that part of the product currently being computed, while the overdots on the next line—the ghosts of the departed underlines—identify the results of that computation:

$$-\, t_2X_6 * -t_1X_5 * W * u_1X_5 * u_2X_6$$

$$\overset{(1)}{=} \quad -t_2X_6 * -t_1X_5 * W * t_1X_5 * -t_1X_5 * u_1X_5 * u_2X_6$$

$$\overset{(2)}{=} \quad -t_2X_6 * \underline{-t_1X_5 * W * t_1X_5} * \underline{-t_1X_5 * u_1X_5} * u_2X_6$$

$$\overset{(3)}{=} \quad -t_2X_6 * \overset{\cdots\cdot}{{}^1W * t_2X_6 * -t_2X_6} * \overset{\cdots\cdots\cdots\cdots}{(u_1 - t_1)X_5} * u_2X_6$$

$$\overset{(4)}{=} \quad \underline{-t_2X_6 * {}^1W * t_2X_6} * -t_2X_6 * (u_1 - t_1)X_5 * u_2X_6$$

$$\overset{(5)}{=} \quad \overset{\cdots\cdot}{{}^2W * -P_6X_6 * P_6X_6} * -t_2X_6 * (u_1 - t_1)X_5 * u_2X_6$$

$$\overset{(6)}{=} \quad \underline{{}^2W * -P_6X_6} * \underline{P_6X_6 * -t_2X_6} * (u_1 - t_1)X_5 * u_2X_6$$

$$\overset{(7)}{=} \quad \overset{\cdots\cdot}{{}^3W} * \overset{\cdots\cdots\cdots\cdots}{(-t_2 + P_6)X_6} * (u_1 - t_1)X_5 * -(-t_2 + P_6)X_6 * (-t_2 + P_6)X_6 * u_2X_6$$

$$\overset{(8)}{=} \quad {}^3W * \underline{(-t_2 + P_6)X_6 * (u_1 - t_1)X_5 * -(-t_2 + P_6)X_6} * \underline{(-t_2 + P_6)X_6 * u_2X_6}$$

$$\overset{(9)}{=} \quad {}^3W * -P_5X_5 * P_5X_5 * \overset{\cdots\cdots\cdots\cdots\cdots}{S_{5,6} * (u_1 - t_1)X_5} * \overset{\cdots\cdots\cdots\cdots\cdots}{(u_2 - t_2 + P_6)X_6}$$

$$\overset{(10)}{=} \quad \underline{{}^3W * -P_5X_5} * \underline{P_5X_5 * S_{5,6} * (u_1 - t_1)X_5} * (u_2 - t_2 + P_6)X_6$$

$$\overset{(11)}{=} \quad \overset{\cdots\cdot}{{}^4W} * \overset{\cdots\cdots\cdots\cdots\cdots\cdots}{S_{5,6} * (u_1 - t_1 + P_5)X_5} * (u_2 - t_2 + P_6)X_6$$

$$\overset{(12)}{=} \underline{{}^4W * {}^1S_{5,6}} * (u_1 - t_1 + P_5)X_5 * (u_2 - t_2 + P_6)X_6$$

$$\overset{(13)}{=} \dot{\dot{\dot{\dot{}}}}\,{}^5W * (u_1 - t_1 + P_5)X_5 * (u_2 - t_2 + P_6)X_6.$$

What is going on here is straightforward enough (albeit somewhat tedious to describe). In equations (1), (2) and (3), we 'home' $-t_1X_5$ across W, producing ${}^1W$, which is now a t_1-dependent element of $\mathfrak{m}_\ell$. In equations (3), (4) and (5), we bring $-t_2X_6$ across ${}^1W$, producing ${}^2W$, a t_1- and t_2-dependent vector which lies *outside* $\mathfrak{m}_\ell$ because $[X_7, X_6] = X_5$. (Recall the non-ideality of $\mathfrak{m}_\ell$.) In equations (5), (6) and (7)), we remove the X_6-direction from ${}^2W$, producing ${}^3W$. In equations (7), (8) and (9), we home $(-t_2 + P_6)X_6$ across $(u_1 - t_1)X_5$, producing $S_{5,6}$, a $t_2(u_1 - t_1)$-dependent vector containing the directions X_1 and X_3. In equations (9), (10) and (11), we remove the X_5-direction from ${}^3W$, producing ${}^4W$. Also in equations (10) and (11), we home P_5X_5 across $S_{5,6}$, leaving $S_{5,6}$ unchanged since $[X_5, X_3] = 0$. Finally, in equations (12) and (13), we multiply ${}^4W$ on the right by $S_{5,6}$, producing ${}^5W$.

Now it is clear, to begin with, that our 'soft' computation has produced a vector of the desired form. Setting $\widetilde{W} = {}^5W$ gives us

$$\begin{aligned}
&- t_2X_6 * -t_1X_5 * W * u_1X_5 * u_2X_6 \\
&= \widetilde{W}(\ell, t_1, t_2, u_1 - t_1) * (u_1 - t_1 + P_5)X_5 * (u_2 - t_2 + P_6)X_6,
\end{aligned}$$

where

$$\widetilde{W}(\ell, t_1, t_2, u_1 - t_1) = (w_1 + P_1)X_1 + \cdots + (w_4 + P_4)X_4 + w_7X_7.$$

But what about the polynomial coefficients? Ignoring most of the minus signs and also the rational coefficients which are not 1, we claim that

$$P_6 = w_7 c(\ell), \text{ the original coefficient of } X_6 \text{ in } W;$$

$$P_5 = w_7 t_2 + w_7^2 c(\ell);$$

$$P_4 = \textstyle\sum(\text{monomials in } w_7, c(\ell), t_1, t_2);$$

$$P_3 = \textstyle\sum(\text{monomials in } w_7, c(\ell), t_1, t_2, u_1 - t_1);$$

$$P_2 = -w_4 t_2 + w_4 w_7 c(\ell) + \textstyle\sum(\text{monomials in } w_7, c(\ell), t_1, t_2);$$

$$P_1 = -w_3 t_2 + w_4 t_1 + w_3 w_7 c(\ell) + w_4 \times \textstyle\sum(\text{monomials in } w_7, t_2, c(\ell)) + \textstyle\sum(\text{monomials in } w_7, c(\ell), t_1, t_2, u_1 - t_1) \; + \; (u_1 - t_1) t_2^2.$$

In 'proof' of this claim, we offer Table 3.3 (on the next page) and some comments. The table contains all possible ways of decomposing the vectors $X_1, \ldots, X_7$ in the strong Malcev basis $\mathcal{B}$ into bracket products of vectors from higher up in the basis. To improve the table's readability, we use a vector's subscript index as its name, and we shall speak accordingly. The zero-th order vectors are just the vectors $1, \ldots, 7$ themselves. The vectors 7 and 6 do not have their own rows in the table because neither can be decomposed into a bracket product— they are the atoms of the algebra. On the other hand, the central vector 1 has thirteen decompositions (thirteen 'names', if you will). The existence of fourth order decompositions of the vectors 1 and 2 indicates that $\mathfrak{n}_7$ is a 5-step Lie algebra.

Zero-th order	First order	Second order	Third order	Fourth order
5	[7,6]	$\cdots$	$\cdots$	$\cdots$
4	[7,5]	[7,[7,6]]	$\cdots$	$\cdots$
3	[7,4]	[7,[7,5]]	[7,[7,[7,6]]]	$\cdots$
	[6,5]	[6,[7,6]]	$\cdots$	$\cdots$
	[7,3]	[7,[7,4]]	[7,[7,[7,5]]]	[7,[7,[7,[7,6]]]]
2	$\cdots$	[7,[6,5]]	[7,[6,[7,6]]]	$\cdots$
	[6,4]	[6,[7,5]]	[6,[7,[7,6]]]	$\cdots$
	[6,3]	[6,[7,4]]	[6,[7,[7,5]]]	[6,[7,[7,[7,6]]]]
	$\cdots$	[6,[6,5]]	[6,[6,[7,6]]]	$\cdots$
1	[4,5]	[4,[7,6]]	$\cdots$	$\cdots$
	$\cdots$	[[7,5],5]	[[7,5],[7,6]]	$\cdots$
	$\cdots$	$\cdots$	[[7,[7,6]],5]	[[7,[7,6]],[7,6]]

Table 3.3

Assuming the dependencies of P_4 to be clear, let us consider P_3. The question is, why does it *not* depend on the variable w_4? Since w_4 is the coefficient of X_4 in W, only the first order bracket [7,4] could conceivably contribute a w_4-term to P_3 (we are looking now at 3's rectangle in the table). But as we saw in our computation of $\widetilde{W}$, $w_4 X_4$ is never bracketed with $w_7 X_7$—hence the absence of w_4-terms from P_3. A similar argument explains why P_2 contains no w_3-terms (look at 2's rectangle in the table and note that only [7,3] could contribute a w_3, etc.). Again, why does P_1 contain only two w_3-terms, to wit, the linear term $-w_3 t_2$ and the non-linear term $w_3 w_7 c(\ell)$? Looking at 1's rectangle, we see that only [6,3] can contribute a term to P_1 containing w_3 as a factor. Now 6 gets bracketed with 3 exactly twice in the computation of $\widetilde{W}$: once as $-t_2X6$ is brought across ${}^1W$, and then again when P_6X_6 is removed from ${}^2W$. The first bracketing produces $-w_3t_2$ and the second produces (up to rational coefficient and sign) $w_3w_7c(\ell)$. So we see how these arguments go.

Now there are certain kinds of terms which, if they occurred in the polynomials $P_1,\ldots,P_4$, would spell disaster for the arguments we plan to make. Specifically, these are terms containing factors of the form $w_4t_1t_2$ or $w_3t_1t_2$ or $w_3w_4t_1t_2$ (or $w_4t_1t_2^2$, etc.). Why do such terms not occur? Why, for example, do no terms containing the factor $w_4t_1t_2$ occur? The answer is not far to seek: such a factor would have to originate from a non-zero bracket product containing X_4, X_5 and X_6 (recalling that $-t_1X_5$ and $-t_2X_6$ are the ultimate sources of factors involving t_1 and t_2). No such product exists, as a rapid perusal of the table reveals. Similar considerations rule out the other dangerous factors.

We hope that enough has been said to convince the reader of the correctness of the listed dependencies of the polynomials $P_1,\ldots,P_4$. In the arguments to follow, the linear terms in P_1 and P_2 will play a crucial role, while the remaining, mostly non-linear terms in all of the polynomials will prove a nuisance. We shall have recourse to the foregoing statements of dependencies in order to show that as we integrate out a particular direction, the inner expression (an integral multiplied by an exponential factor) does not vary in undesirable ways. So let us return to the kernel K_φ^ℓ.

Setting $a_1 = u_1 - t_1 + P_5$ and $a_2 = u_2 - t_2 + P_6$, and noting that $\ell(W) = \ell_1 w_1 + \ell_2 w_2 + \ell_7 w_7$, we now find that

$$\begin{aligned}
&K_\varphi(t_1, t_2, u_1, u_2, \ell) \\
&= \int_{\mathfrak{m}_\ell} \tilde{\varphi}\big(\widetilde{W} * a_1 X_5 * a_2 X_6\big)\ e^{2\pi i\, \ell\left(W\right)}\ dW \\
&= \int_{\mathbb{R}^5} \tilde{\varphi}\big(w_1 + P_1, \ldots, w_4 + P_4, w_7\,;\, a_1, a_2\big) \times \\
&\qquad\qquad e^{2\pi i\left(\ell_1 w_1 + \ell_2 w_2 + \ell_7 w_7\right)}\ dw_1\ dw_2\ dw_3\ dw_4\ dw_7 \\
&\overset{(3)}{=} \int_{\mathbb{R}} \Bigg(\int_{\mathbb{R}^4} \tilde{\varphi}\big(w_1 + P_1, \ldots, w_4 + P_4, w_7\,;\, a_1, a_2\big) \times \\
&\qquad\qquad e^{2\pi i\left(\ell_1 w_1 + \ell_2 w_2\right)}\ dw_1\ dw_2\ dw_3\ dw_4 \Bigg)\ e^{2\pi i\,(\ell_7 w_7)}\ dw_7 \\
&= \int_{\mathbb{R}} {}^1K_{\tilde{\varphi}}\big(\ell_1, \ell_2, t_2, t_1, w_7\,;\, a_1, a_2\big)\ e^{2\pi i\,(\ell_7 w_7)}\ dw_7,
\end{aligned}$$

where ${}^1K_{\tilde{\varphi}}\big(\ell_1, \ell_2, t_2, t_1, w_7\,;\, a_1, a_2\big)$ refers to the inner 4-dimensional integral on the right-hand side of equation (3).

We observe that this abridged kernel ${}^1K_{\tilde{\varphi}}\big(\ell_1, \ell_2, t_1, t_2, w_7\,;\, a_1, a_2\big)$ is *independent* of ℓ_7 and compactly supported in the variable w_7. Because of these facts, the Paley-Wiener theorem implies that the integral

$$(*) \qquad \int_{\mathbb{R}} {}^1K_{\tilde{\varphi}}\big(\ell_1, \ell_2, t_2, t_1, w_7\,;\, a_1, a_2\big)\ e^{2\pi i\,(\ell_7 w_7)}\ dw_7,$$

defines a function of ℓ_7 (a partial Fourier transform of the abridged kernel) which extends to an entire function on $\mathbb{C}$.

By hypothesis, the integral $(*)$ is zero for all $t_1, t_2, u_1, u_2 \in \mathbb{R}$ and for all $\ell \in E'$. Since we have put $a_1 = u_1 - t_1 + P_5$ and $a_2 = u_2 - t_2 + P_6$, we note that a_1 and a_2 can be fixed arbitrarily by letting $u_1 = a_1 + t_1 - P_5$ and $u_2 = a_2 + t_2 - P_6$, respectively. Hence, the integral $(*)$ is zero for all $t_1, t_2, a_1, a_2 \in \mathbb{R}$ and for all $\ell \in E'$.

Let us now consider ℓ to be the ordered triple (ℓ_1, ℓ_2, ℓ_7). Because E' has positive 3-dimensional measure, there exists a set $E'_{1,2}$ of positive 2-dimensional measure such that for each pair $(\ell_1, \ell_2) \in E'_{1,2}$, the triple (ℓ_1, ℓ_2, ℓ_7) is an element of E' for all ℓ_7 in a set $E'_7(\ell_1, \ell_2)$ of positive 1-dimensional measure. (This is, as before, a consequence of Fubini's theorem.)

Fix $t_1, t_2, a_1, a_2 \in \mathbb{R}$ arbitrarily and fix the pair (ℓ_1, ℓ_2) arbitrarily in the set $E'_{1,2}$. Then for all ℓ_7 in the set $E'_7(\ell_1, \ell_2)$, the integral $(*)$ vanishes. Its possession of an entire extension then implies its vanishing for *all* $\ell_7 \in \mathbb{R}$. But this vanishing for all ℓ_7 implies, in turn, that

$$\begin{aligned}
0 &= {}^1K_{\tilde{\varphi}}(\ell_1, \ell_2, t_2, t_1, w_7\,;\, a_1, a_2) \\
&\overset{(2)}{=} \int_{\mathbb{R}^4} \tilde{\varphi}\big(w_1 + P_1,\, w_2 + P_2,\, w_3 + P_3,\, w_4 + P_4,\, w_7\,;\, a_1,\, a_2\big) \times \\
&\qquad\qquad\qquad\qquad e^{2\pi i \left(\ell_1 w_1 + \ell_2 w_2\right)}\, dw_1 dw_2 dw_3 dw_4
\end{aligned}$$

for each fixed choice of $t_1, t_2, a_1, a_2 \in \mathbb{R}$ and, given any fixed pair $(\ell_1, \ell_2) \in E'_{1,2}$, for each fixed $w_7 \in \mathbb{R}$.

The integral in equation (2) is taken over the four variables $w_1, \ldots, w_4$, but the exponential factor contains only the variables w_1 and w_2. The missing w_3 and w_4 variables must in fact be carried upstairs from the argument of the function $\tilde{\varphi}$ by a change of variables. First, we send $w_4 \longmapsto w_4 - P_4$. We saw above that P_3 does not depend on w_4; hence this translation alters only P_1 and P_2, which become

$$\begin{aligned} {}^1P_2 = &\ -w_4t_2 + w_4w_7c(\ell) + \underline{\textstyle\sum(\text{monomials in } w_7, c(\ell), t_1, t_2)} \\ &+ \underline{P_4t_2} - \underline{P_4w_7c(\ell)}, \end{aligned}$$

$$\begin{aligned} {}^1P_1 = &\ -w_3t_2 + w_3w_7c(\ell) + w_4t_1 + w_4 \times \textstyle\sum(\text{monomials in } w_7, t_2, c(\ell)) \\ &+ \underline{\textstyle\sum(\text{monomials in } w_7, c(\ell), t_1, t_2, u_1 - t_1)} + \underline{(u_1 - t_1)t_2^2} \\ &\underline{-P_4t_1} - \underline{P_4 \times \textstyle\sum(\text{monomials in } w_7, t_2, c(\ell))}. \end{aligned}$$

The underlined terms of these two polynomials do not depend on the remaining variables of integration. Thus, sending

$$w_1 \longmapsto w_1 - (\text{the underlined terms in } {}^1P_1)$$

produces an exponential factor which can be removed from the integrand and then cancelled (since the integral in equation (2) vanishes for the values of the variables with which we are concerned). A similar thing happens if we send

$$w_2 \longmapsto w_2 - (\text{the underlined terms in } {}^1P_2).$$

So let us consider these two additional translations and subsequent cancellations as having been made, producing the more manageable polynomials

$$\begin{aligned} {}^2P_2 &= -w_4t_2 + w_4w_7c(\ell) \\ {}^2P_1 &= -w_3t_2 + w_3w_7c(\ell) + w_4t_1 + w_4 \times \textstyle\sum(\text{monomials in } w_7, t_2, c(\ell)). \end{aligned}$$

If we now also send $w_3 \longmapsto w_3 - P_3$, we find that ${}^2P_2$ does not change, but

$$
\begin{aligned}
{}^3P_1 = \; & -w_3t_2 + w_3w_7c(\ell) + w_4t_1 + w_4 \times \textstyle\sum(\text{monomials in } w_7, t_2, c(\ell)) \\
& + \underline{P_3t_2} - \underline{P_3w_7c(\ell)}.
\end{aligned}
$$

Again, the underlined terms do not depend on $w_1, \ldots, w_4$. Repeating the procedure of a moment ago, we send

$$w_1 \longmapsto w_1 - (\text{the underlined terms in } {}^3P_1)$$

and cancel the resulting exponential factor, producing the polynomial

$${}^4P_1 = \; -w_3t_2 + w_3w_7c(\ell) + w_4t_1 + w_4 \times \textstyle\sum(\text{monomials in } w_7, t_2, c(\ell)),$$

which is identical with ${}^2P_2$.

Before we can continue our integrations, we must finish the task of bringing the variables w_3 and w_4 up into the exponential $e^{2\pi i\,(\ell_1w_1+\ell_2w_2)}$. To this end, we make our final changes of variables, sending $w_1 \longmapsto w_1 - {}^4P_1$ and $w_2 \longmapsto w_2 - {}^2P_2$. The integral that we last saw in equation (2) now has the form

$$
\begin{aligned}
& \int_{\mathbb{R}^4} \tilde{\varphi}\big(w_1, w_2, w_3, w_4, w_7\,;\, a_1, a_2\big) e^{2\pi i Q} e^{2\pi i\left(\ell_1w_1+\ell_2w_2+\tilde{\ell}_3w_3+\tilde{\ell}_4w_4\right)}\; dw_1dw_2dw_3dw_4 \\
& = \int_{\mathbb{R}} \Bigg(e^{2\pi i Q_4} \int_{\mathbb{R}^3} \tilde{\varphi}\big(w_1, w_2, w_3, w_4, w_7\,;\, a_1, a_2\big)\; e^{2\pi i Q_3} \times \\
& \qquad\qquad\qquad e^{2\pi i\left(\ell_1w_1+\ell_2w_2+\tilde{\ell}_3w_3\right)}\; dw_1dw_2dw_3 \Bigg)\; e^{2\pi i(\tilde{\ell}_4w_4)}\; dw_4,
\end{aligned}
$$

where we have introduced the abbreviations:

$$\tilde{\ell}_3 = t_2\ell_1$$

$$\tilde{\ell}_4 = t_2\ell_2 - t_1\ell_1$$

$$\begin{aligned} Q &= Q_3 + Q_4 \\ &= w_3\big(-w_7\ell_1 c(\ell)\big) + w_4\big(-w_7\ell_2 c(\ell) + (\text{terms in } w_7, t_2, c(\ell))\big). \end{aligned}$$

The polynomial Q contains the remnants of the non-linear terms from the argument of $\tilde{\varphi}$, Q_3 containing the single w_3-term and Q_4 containing the w_4-terms. $\tilde{\ell}_3$ and $\tilde{\ell}_4$ are now the third and fourth components, respectively, of the strongly generic parametrizing functionals in E'. For future reference, we point out that the variable t_1 does not appear in Q, and the variable t_2 appears only in Q_3.

Consider now the linear system

$$\begin{cases} \tilde{\ell}_3 = t_2\ell_1 \\ \tilde{\ell}_4 = t_2\ell_2 - t_1\ell_1 \,. \end{cases}$$

Since ℓ is strongly generic, which in this example means that $\ell_1 \neq 0$, we may solve for t_2 in the top equation of this system, and then substitute in the bottom equation, obtaining

$$\tilde{\ell}_4 = \frac{\tilde{\ell}_3}{\ell_1}\ell_2 - t_1\ell_1 = -\tilde{\ell}_3 c(\ell) - t_1\ell_1 .$$

If we fix the pair $(\ell_1, \ell_2) \in E'_{1,2}$ and also fix $t_2 \in \mathbb{R}$, we see that $\tilde{\ell}_3$ is also fixed, but $\tilde{\ell}_4$ may be varied *ad libitum* by varying t_1.

What this means is that if we write

$$= \int_{\mathbb{R}} \left(e^{2\pi i Q_4} \int_{\mathbb{R}^3} \tilde{\varphi}(w_1, w_2, w_3, w_4, w_7\,;\, a_1, a_2)\; e^{2\pi i Q_3} \times \right.$$

$$\left. e^{2\pi i\left(\ell_1 w_1 + \ell_2 w_2 + \tilde{\ell}_3 w_3\right)}\; dw_1 dw_2 dw_3 \right) e^{2\pi i(\tilde{\ell}_4 w_4)}\; dw_4$$

$$= \int_{\mathbb{R}} {}^2K_{\tilde{\varphi}}\big(\ell_1,\, \ell_2,\, \tilde{\ell}_3,\, w_4,\, w_7\,;\, a_1,\, a_2\big)\; e^{2\pi i\, \tilde{\ell}_4 w_4}\; dw_4,$$

where ${}^2K_{\tilde{\varphi}}\big(\ell_1,\, \ell_2,\, \tilde{\ell}_3,\, w_4,\, w_7\,;\, a_1,\, a_2\big)$ denotes the parenthesized expression on the right-hand side of the first equation, then this twice-abridged kernel is *independent* of $\tilde{\ell}_4$ (which is another way of saying that varying t_1 does not shift the kernel), and it is also compactly supported in the variable w_4. The Paley-Wiener theorem again tells us that the integral

$$(**) \qquad \int_{\mathbb{R}} {}^2K_{\tilde{\varphi}}\big(\ell_1,\, \ell_2,\, \tilde{\ell}_3,\, w_4,\, w_7\,;\, a_1,\, a_2\big)\; e^{2\pi i\, \tilde{\ell}_4 w_4}\; dw_4$$

defines a function of $\tilde{\ell}_4$ (a partial Fourier transform of the twice-abridged kernel) which extends to an entire function on $\mathbb{C}$.

Having travelled this route before, we know what comes next. For each fixed pair $(\ell_1, \ell_2) \in E'_{1,2}$, and for each fixed choice of $\tilde{\ell}_3, w_7, a_1, a_2 \in \mathbb{R}$, the integral $(**)$ vanishes for all $\tilde{\ell}_4 \in \mathbb{R}$. The existence of an entire extension of $(**)$ then implies the vanishing of ${}^2K_{\tilde{\varphi}}\big(\ell_1,\, \ell_2,\, \tilde{\ell}_3,\, w_4,\, w_7\,;\, a_1,\, a_2\big)$ for all $w_4 \in \mathbb{R}$ (under the same hypotheses). We remark that $e^{-2\pi i Q_4}\big({}^2K_{\tilde{\varphi}}\big)$ also vanishes, and so we cancel the exponential containing Q_4.

One more rewrite of the remaining integral puts us where we want to be:

$$\int\limits_{\mathbb{R}^3} \tilde{\varphi}\big(w_1, w_2, w_3, w_4, w_7\,;\, a_1,\, a_2\big)\ e^{2\pi i Q}\ e^{2\pi i\left(\ell_1 w_1+\ell_2 w_2+\tilde{\ell}_3 w_3\right)}\ dw_1 dw_2 dw_3$$

$$= \int\limits_{\mathbb{R}} \Bigg(e^{2\pi i Q_3} \int\limits_{\mathbb{R}^2} \tilde{\varphi}\big(w_1, w_2, w_3, w_4, w_7\,;\, a_1,\, a_2\big) \times$$

$$e^{2\pi i\left(\ell_1 w_1+\ell_2 w_2\right)}\ dw_1 dw_2 \Bigg)\ e^{2\pi i\left(\tilde{\ell}_3 w_3\right)} dw_3.$$

It is clear that the parenthesized expression ${}^3K_{\tilde{\varphi}}$ (as we should call it) is independent of $\tilde{\ell}_3$, that the whole integral, considered as a function of $\tilde{\ell}_3$, has an entire extension to $\mathbb{C}$ (Paley-Wiener), and that under hypotheses identical to those above (supplemented by the additional freedom in w_4), the vanishing of the whole integral for all $\tilde{\ell}_3 \in \mathbb{R}$ implies the vanishing of $e^{-2\pi i Q_3}\left({}^3K_{\tilde{\varphi}}\right)$ for each fixed pair $(\ell_1, \ell_2) \in E'_{1,2}$ and for each choice of $w_3, w_4, w_7, a_1, a_2 \in \mathbb{R}$.

Under these last hypotheses, we now have that

$$0 = \int\limits_{\mathbb{R}^2} \tilde{\varphi}\big(w_1, w_2, w_3, w_4, w_7\,;\, a_1,\, a_2\big)\ e^{2\pi i\left(\ell_1 w_1+\ell_2 w_2\right)}\ dw_1 dw_2.$$

Since this integral is the partial Fourier transform of $\tilde{\varphi}$ in the central variables w_1 and w_2, and since it vanishes for all pairs (ℓ_1, ℓ_2) in a set of positive 2-dimensional measure, it vanishes for all pairs (ℓ_1, ℓ_2). Hence the function $\tilde{\varphi} = \varphi \circ \exp$ on $\mathfrak{n}_7$ vanishes for almost all $w_1, w_2 \in \mathbb{R}$ and for all $w_3, w_4, w_7, a_1, a_2 \in \mathbb{R}$. So we see, finally, that $\varphi = 0$ almost everywhere on N_7, as desired. $\square$

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Vita

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